

THE EFFECT OF NOISE IN
CHAOTIC GALACTIC POTENTIALS

By

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To My Parents

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The fact that 'realistic' galactic potentials are not necessarily integrable implies that a large number of stars should be expected to follow stochastic trajectories. Orbits of stars in an elliptical galaxy are assumed to be affected by both the smooth mean field and internal fluctuations associated with discreteness effects. The interplay of these two effects is studied in the context of the gravitational N -body problem. These studies indicate that both the mean field and discreteness effects have an effect on the instability of the trajectories. These discreteness effects are also modelled as noise, perturbing trajectories in a nonintegrable potential. The conclusion here is that perturbations in the phase space variables grow exponentially on a time scale associated with the instability of the deterministic trajectories, even though the time scale associated with changes in collisionless invariants remains the conventional relaxation time. Because physically relevant nonintegrable potentials are typically non-hyperbolic, the standard argument that shadowing implies that noisy orbits behave like deterministic orbits does not hold. A nonhyperbolic shadowing theorem is used to determine bounds on the effect that noise has in such a potential. This noise is interpreted to be due to discreteness effects caused

by close encounters. Shadowing times are calculated for both maps and differential equations, yielding the following conclusions. First, shadowing times for such systems tend towards a roughly log-normal distribution. These distributions appear to exhibit a fractal dependence on initial conditions. It has been conjectured previously that the shadowing times are expected to be proportional to $1/\sqrt{\delta}$, where δ is the amplitude of the noise. It is found that, for low noise levels in Hamiltonian differential equations, the shadowing times are considerably shorter than this conjecture would indicate. Finally, the locations where shadowing breaks down appear to be correlated with the location of KAM tori and cantori of Hamiltonian systems.

CHAPTER 1 INTRODUCTION TO GALACTIC DYNAMICS

Galaxies are systems which consist of a large number of stars, gas, and dust. Typically, the number of stars ranges from 10^{10} to 10^{12} . Despite the complicated interactions among the various components of the innumerable galaxies, they can generally be classified into spirals and ellipticals, with a few being irregular. Elliptical galaxies are observed to be virtually dust-free, and hence are usually modelled as a collection of stars interacting via gravitational forces, described by Newton's Law of Gravitation. As such, it is tempting to describe the dynamics of these galactic systems using results from celestial mechanics, classical statistical mechanics, and plasma physics. However, there are difficulties with each of these approaches, and hence galactic dynamics should be considered a branch of theoretical physics in its own right.

The analogy with calculations in celestial mechanics, which describe the motions of objects in the solar system, may appear useful since we are considering gravitationally bound systems. However, they are based on perturbation expansions which are based on a hierarchy of mass scales that do not converge in the case of stellar systems. Hence any mathematical formalism based on celestial mechanics is essentially of no use in the study of stellar systems.

The large number of particles in galactic systems requires that orbits in a galactic system be treated statistically. However, the long-range nature of

gravitational forces means that they must be treated differently than the short-range forces associated, e.g., with the motions of gas molecules confined to a box. In a later chapter, we will see in more detail why orbits in a gravitational system do not obey Boltzmann's ergodic hypothesis, which is the foundation of equilibrium statistical mechanics.

Finally, plasma physics, which also involves the study of a large number of particles interacting via long-range forces, in this case electrostatic, requires many of the same mathematical tools as galactic dynamics. However, the existence of both positive and negative charges leads to screened potentials and static, homogeneous equilibria, which are absent in galactic systems. Nevertheless, the collisional Boltzmann equation, primarily used in plasma physics, is frequently relied upon in the study of galactic systems and, in fact, Jeans formulated the equation in the context of gravitational systems before it was used for plasma physics.

The dynamics of stars in galactic systems may be extremely complicated, and any reasonable model of a galactic potential must take into account the fact that individual stellar trajectories may not, in generic situations, be calculated analytically. Nevertheless, methods exist that make a statistical treatment of stellar dynamics feasible. Standard methods include N-body calculations and the study of the collisionless Boltzmann equation. While both of these methods have their shortcomings, to be described later, they do have the ability to yield useful statistical results.

It is the purpose of this chapter to outline various results of galactic dynamics, found via standard methods, and to motivate the idea that the modern techniques of dynamical systems theory are necessary. To begin with, the

standard paradigm of decoupling collisional effects from bulk mean field effects will be discussed, leading to the conventional calculation of a characteristic relaxation time for stellar systems. Then, some discussion of how collisional effects should be incorporated will be included. After this, I will attempt to convince the reader that the apparent existence of nonintegrable galactic potentials makes modern dynamical systems techniques of use for galactic systems.

It should be noted that the arguments presented in this chapter are not rigorous. The field of galactic dynamics draws upon results from other disciplines, for example, plasma physics, which are based on assumptions which may not be suitable for galactic systems. Therefore, the bulk of this chapter should be taken as an attempt to briefly discuss the conventional wisdom in the field, and is not meant to imply that the author concurs with the beliefs presented here.

1.1 Characteristic Time Scales in Stellar Dynamics

1.1.1 The Gravo-thermal Catastrophe

When one desires to calculate the dynamical properties of a given physical system statistically, it is usually necessary to determine the proper time scales over which the system evolves. One is often interested, for example, in the approach to equilibrium. However, for self-gravitating systems, care must be taken in defining 'equilibrium,' since a thermal equilibrium does not exist for such systems.

This is easily seen, as discussed in detail, e.g., by Lynden-Bell and Lynden-Bell [1]. Let us consider the following heuristic argument. If the stars in a galaxy were considered to be an ideal gas, the theorem of equipartition of

energy would be expected to hold, relating the mean-square velocity of the particles $\overline{v^2}$ to the temperature of the system T :

$$\frac{1}{2}m\overline{v^2} = \frac{3}{2}k_B T, \quad (1.1)$$

where m is the stellar mass and k_B is Boltzmann's constant. The Virial Theorem, which states that for an isolated system, neglecting boundaries, the total energy is equal to the negative of the total kinetic energy, allows us to write the total energy of a galactic system with N particles in terms of the temperature of the system,

$$E = -\frac{3}{2}Nk_B T. \quad (1.2)$$

The heat capacity of the system is then calculated to be negative:

$$C_V \equiv \frac{dE}{dT} = -\frac{3}{2}Nk_B. \quad (1.3)$$

This apparently paradoxical result of a negative heat capacity is a consequence of the fact that, in general, gravitational systems cannot be described in terms of a canonical ensemble, and cannot be considered to be in a thermal equilibrium. This is true for *any* bound, finite, unconstrained, gravitating system, and is responsible for the so-called *gravothermal catastrophe*: a thermally isolated sphere is unstable to perturbations. If the central core of the sphere becomes hotter than the periphery, it will continue to become hotter and denser, leading to a runaway effect. Thus one cannot, in a true sense, speak of a thermal equilibrium for gravitational systems. Instead, one must be content to consider gravitational systems that approach, for example, a dynamical equilibrium. It is this sort of 'quasi-equilibrium' that will be considered when we wish to discuss the relevant time scales of the gravitational system.

1.1.2 The Dynamical Time

Now that we see that the concept of ‘equilibrium’ is an elusive quantity, we must be careful with determining the time scales associated with an approach to equilibrium. Let us consider that the system approaches some ‘quasi-stationary state’ due to the action of the long-range gravitational forces acting as a smooth mean field. The system can then be considered to be in a virial equilibrium, so the kinetic energy of the system should be comparable in magnitude to the potential energy, and

$$\bar{v}^2 \sim \frac{GNm}{R}. \quad (1.4)$$

Since the ‘collective’ force acting on a test particle of unit mass will have a typical magnitude of $\bar{F}_{tot} \sim \frac{GNm}{R^2}$, we can determine this characteristic dynamic time scale to be $\tau_d \sim \frac{\bar{v}}{\bar{F}_{tot}}$, which is the crossing time, $\tau_c \sim R/\bar{v}$, where R is the characteristic length scale of the system and $\bar{v}$ is the stellar root-mean-square velocity.

1.1.3 The Binary Relaxation Time

Of course, one has to realize that not only the bulk mean field, but also encounters with nearby stars can affect the stellar trajectories. Generally, one may wish to calculate the ‘relaxation time,’ the characteristic time for which a star would forget its initial conditions via interactions with nearby stars. Specifically, one can calculate the binary relaxation time, a.k.a. gravitational Rutherford scattering, which is usually defined to be the time required for the velocity of a test star to change appreciably due to graininess effects as it crosses the system, or

$$\tau_r = \frac{v^2}{\Delta v_1^2} \tau_c, \quad (1.5)$$

where $\Delta v_{\perp}^2$ is the integrated deflection in the velocity due to the rest of the N stars. If we again utilize the virial theorem, it can be shown that (see, e.g. [2])

$$\tau_r \approx 0.1N/\ln \Lambda, \quad (1.6)$$

where Λ is the ratio of the impact parameters $\frac{b_{max}}{b_{min}}$. In plasma physics, b_{max} is the Debye cutoff, which does not exist in gravitational physics. Instead, b_{max} is considered to be about the size of the system. Exactly how one defines this value is open to debate, however, in any event $\ln \Lambda \sim \ln N$. The determination of this binary relaxation time, however, is not unique. τ_r may also be interpreted as the time over which appreciable changes in energy occur, for example, or the time required for equipartition of energy to occur. The standard analysis neglects the movement of the other stars as well. In any event, if the masses of the stars are all comparable, these differences do not have much of an effect on the value of τ_r . A typical value of τ_r for an elliptical galaxy would be 10^{14} - 10^{16} years.

Numerical values for these characteristic times are easy to determine. For a typical elliptical galaxy of $N \approx 10^{11}$ stars, the dynamical time is of the order 10^8 years, whereas the binary relaxation time, 10^{14} years, is much longer than the age of the universe. Thus, from this analysis, it would appear that, over relevant time scales, binary encounters are unimportant. However, it must be noted that this derivation of the binary relaxation time is based on the assumption that, without encounters, the stellar trajectories follow a smooth, rectilinear trajectory, i.e. the trajectories are not exponentially unstable.

1.1.4 Collective Relaxation

Perhaps a more relevant concept is the idea of collective relaxation. Here, we are assuming that the system exhibits sensitive dependence on initial conditions, which is certainly true for a large choice of potentials. In general, this sensitivity is exponential, causing trajectories initially a distance $\mathbf{x}(0)$ apart to be, after a time t , a distance $e^{t/\tau_c} \mathbf{x}(0)$ apart. This time scale, the collective relaxation time scale, can be estimated numerically (see Chapter 3) or analytically. The mathematical formulation of the problem was discussed in a gravitational setting by Kandrup [3], who viewed the dynamics of the stellar trajectories as a geodesic flow on a $3N$ -dimensional manifold with average negative curvature, and concluded that this collective relaxation should be manifest on a time scale $\sim \tau_c$. This conclusion was also reached by Goodman, Heggie, and Hut [4], using a very different approach.

It should be stressed again at this point that dynamically galaxies are relatively young objects, having lived for typically no more than 100 crossing times, and not nearly old enough for binary relaxation mechanisms to have had an effect. This latter point is especially meaningful when one considers the effect that nearby stars may have on the divergence of trajectories in a stellar system. It may be true that the classical binary relaxation time proceeds on a time scale much too long to be considered for galactic systems. However, this does not a priori imply that nearby encounters are irrelevant, if one appropriately considers the collective relaxation of the system. Nevertheless, it is commonly assumed that, on appropriate time scales, nearby stellar encounters are unimportant, an assumption that would make analytical calculations easier. This is the idea explored in the next section.

1.2 The Collisionless Boltzmann Equation

It is common to assume that, if a gravitational system is large, the effect of close encounters between nearby stars will be small compared to the overall mean field. That this be so can presumably be determined from the following argument. Let us assume here that all stars have the same mass. The average distance between stars in a system of size R is $R/N^{1/3}$, yielding a potential energy between the two stars $E_2 \simeq GmN^{1/3}/R$ per star. Thus, the ratio between the average potential energy between two stars and the total potential energy per star, $E_T = GmN/R$, is

$$E_2/E_T \simeq N^{-2/3}.$$

From this, it is frequently believed that close encounters have a negligible effect on the relaxation of galactic systems.

With this in mind, and realizing that gravitational systems are far from being in thermodynamic equilibrium, it is standard to describe the distribution of stars in phase space using kinetic equations. The state of the system can be completely described by the N -particle distribution function $f^{(N)}$, which obeys Liouville's equation,

$$\frac{df^{(N)}}{dt} = 0, \quad (1.7)$$

where $\frac{d}{dt}$ is the convective derivative in the N -dimensional phase space. The probability that a star will be found in a certain cell in phase space is given by

$$f^{(N)}(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{p}_1, \dots, \mathbf{p}_N, t) d^3x_1 \dots d^3x_N d^3p_1 \dots d^3p_N. \quad (1.8)$$

It should be apparent that $f^{(N)}$ is of little use when N is large, since it is such a complex and unwieldy object. Instead of working directly with $f^{(N)}$,

we can define a one-particle distribution function by

$$f^{(1)}(\mathbf{x}_1, \mathbf{p}_1, t) = \int f^{(N)}(\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{p}_1, \dots, \mathbf{p}_N, t) d^3x_2 \dots d^3x_N d^3p_2 \dots d^3p_N, \quad (1.9)$$

where

$$f^{(1)}(\mathbf{x}, \mathbf{p}, t) d^3x d^3p \quad (1.10)$$

is the probability of finding a single particle in the phase space volume $d^3x d^3p$ around the point $(\mathbf{x}, \mathbf{p})$.

We can as well define a two-particle distribution function as

$$f^{(2)}(\mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2, t) = f^{(1)}(\mathbf{x}_1, \mathbf{p}_1, t) f^{(1)}(\mathbf{x}_2, \mathbf{p}_2, t) [1 + g(\mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2, t)], \quad (1.11)$$

where g is a two-particle correlation function. If we then make the claim that $g = 0$, which is exact for noninteracting particles and a good approximation for weakly interacting particles, we obtain for the one-particle distribution function (called f for brevity)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{p}} = 0. \quad (1.12)$$

Equation (1.12) is called the collisionless Boltzmann equation, which is also known as the electrostatic Vlasov equation in plasma physics. In stellar dynamics, we are considering only the gravitational force in $-\nabla \Phi$ and the momentum $\mathbf{p}$, whereas, in general, electric and magnetic forces are included as well. It should also be noted that the gravitational potential Φ depends on the mean field, and can be self-consistently determined by the Poisson equation

$$\nabla^2 \Phi = -4\pi G m \int dv f(\mathbf{x}, \mathbf{v}, t). \quad (1.13)$$

Unlike the full N -dimensional Liouville equation, the collisionless Boltzmann equation is simple enough to be tractable. One equilibrium solution of this equation is simply a Maxwell-Boltzmann distribution, never fully realized for gravitational systems due, to some extent, to escapes and to the gravothermal catastrophe, which nevertheless eventually relax to a quasi-Maxwellian equilibrium. As it has been supposedly determined that this relaxation could not be due to close encounters, another relaxation process must have taken place for this to occur, since galaxies are apparently observed to have relaxed. This collective relaxation has been called ‘violent relaxation’ by Lynden-Bell [5], for which the distribution function f changes rapidly, on the order of a dynamical time τ_d .

The existence of a violent relaxation, however, does not guarantee a thermodynamic quasiequilibrium. During the course of the relaxation, the distribution of particles gets scrambled. It is then useful to consider a coarse-grained distribution function, which averages over phase cells. This coarse-grained distribution function then obeys an exclusion principle akin to Fermi-Dirac statistics, since Liouville’s theorem disallows degeneracy of phase elements. However, this new distribution function does not obey the collisionless Boltzmann equation, and hence its equilibrium solution is not necessarily Maxwellian.

1.3 Collisional Stellar Dynamics

1.3.1 The Collisional Boltzmann Equation

Of course the collisionless Boltzmann equation is not valid whenever close encounters are important. In this case, it is necessary to include a ‘collision term’ in equation (1.12), noting that ‘collision’ does not imply any actual

physical contact, rather, that the stars are close enough for graininess to be relevant. Thus the (collisional) Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - \nabla \phi \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}. \quad (1.14)$$

The right-hand side of equation (1.14) includes only instantaneous binary collisions, which is a good approximation for dilute gasses with short range forces, and plasmas, for which the long range interaction is screened. Though the assumptions are suspect, it is still generally believed to be relevant for gravitational systems when N is large.

The collision term is nonzero whenever the two-particle direct correlation function is nonzero; loosely speaking, this is the more-or-less obvious statement that correlations between two particles are due to a close encounter between them. When this is true, the collision term in the Boltzmann equation, which describes the evolution of the one-particle distribution function, is written in terms of the two-particle distribution function, and, in general, the equation for the evolution of the n -particle distribution function contains an $n + 1$ -particle distribution function. This hierarchy of equations is known as the BBGKY hierarchy. The Boltzmann equation, the first of the equations in the hierarchy, is obtained in the limit of dilute gasses, when only binary collisions need be accounted for.

The first two equations of the BBGKY hierarchy are relevant and useful for plasma physics, due to the short-range nature of the correlations from the Debye screening. However, there is a difficulty with using these equations for gravitational systems. Due to the long-range nature of the force, the mean field can never be neglected.

The question remains how to treat the collision term in the Boltzmann equation. This term can be approximated if we consider only weak encounters, that is, encounters for which the velocity change due to the encounter is small compared to the original velocity of the star. This is a reasonable approximation for galactic systems, since for such systems encounters which yield a drastic velocity change are extremely rare. We can then expand the collision term in a Taylor series for the phase space variables of the star. Truncating the series after the second-order terms, we have an approximate solution for the collision term

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = - \sum_{i=1}^6 \frac{\partial}{\partial w_i} [f(\mathbf{w}) D(\Delta w_i)] + \frac{1}{2} \sum_{i,j=1}^6 \frac{\partial^2}{\partial w_i \partial w_j} [f(\mathbf{w}) D(\Delta w_i \Delta w_j)], \quad (1.15)$$

where w_i are the phase space variables.

Equation (1.15) is the Fokker-Planck approximation, and $D(\Delta w_i)$ and $D(\Delta w_i \Delta w_j)$, which are functionals of f , are known as the diffusion coefficients. The difficulty now lies in calculating the diffusion coefficients from the one-particle distribution function. The equation is similar to the actual Fokker-Planck equation, a master equation for the probability $P(y, t)$

$$\frac{\partial P(y, t)}{\partial t} = - \frac{\partial}{\partial y} A(y) P(y, t) + \frac{1}{2} \frac{\partial^2}{\partial y^2} B(y) P(y, t). \quad (1.16)$$

1.3.2 The Langevin Equation

Collisional effects can be treated with an entirely different approach than through the Boltzmann equation as considered heretofore. Consider the velocity of a free particle (i.e. in the absence of external forces) subject to random fluctuations. The equation of motion of the particle can then be written

$$\dot{\mathbf{v}} = -\gamma \mathbf{v} + L(t). \quad (1.17)$$

This is the classical picture of Brownian motion, and equation (1.17) is known as the Langevin equation. The force on the right-hand side of the equation consists of a drag force which is linear in the velocity, and a random external force, which is due to the collisions of the particle with other particles. $L(t)$ is considered to be a stochastic process, defined so that the average $\langle L(t) \rangle = 0$. Usually when dealing with rapidly varying stochastic forces, i.e. the random bombardment of a Brownian particle by the atoms in the surrounding fluid, the autocorrelation function is expressed as a delta function,

$$\langle L(t)L(t') \rangle = \Gamma \delta(t - t'), \quad (1.18)$$

where Γ is a constant.

By writing equation (1.17), we have made the assumption that the force acting on a particle can be decomposed into two independent parts, one being a smooth friction, and the other due to the discontinuities of the matter distribution. Two assumptions have been made about the fluctuations: $L(t)$ is assumed to be independent of the velocity $\mathbf{v}$, and $L(t)$ is assumed to vary extremely rapidly compared to variations in $\mathbf{v}$.

The above assumptions are certainly valid for the case of Brownian motion. In stellar dynamics, one may estimate the the average period of the fluctuation of the influence of the immediate neighborhood of stars to be $T \simeq D/\bar{v}$, where $\bar{v}$ is the root mean square relative velocity between the two stars, and D is the average distance between nearest neighbors, which can be calculated to be $D \simeq 0.55396n^{-1/3}$, where n is the average number of stars per unit volume. If we consider the effect of such fluctuations on the sun, for example, we have $D \sim 3$ parsecs, and $\bar{v} \sim 50$ km/sec. Thus we obtain $T \simeq 6 \times 10^4$ years.

Comparing this to the period of galactic rotation, which is about 2×10^8 years, we see that the fluctuating force does indeed vary sufficiently rapidly.

The postulates of the Langevin equation only specify the first two moments, saying nothing of the higher moments of $L(t)$. If we include an additional postulate that $L(t)$ is Gaussian, i.e. all higher cumulants are zero, $L(t)$ is called *Gaussian white noise*. The solution of $v(t)$ can then be shown to be Gaussian. One can then conclude that the Langevin equation with Gaussian white noise, determined by equation (1.17) and equation (1.18), is equivalent to the Fokker-Planck equation

$$\frac{\partial P(v, t)}{\partial t} = \gamma \frac{\partial}{\partial v} v P + \frac{\Gamma}{2} \frac{\partial^2 P}{\partial v^2}, \quad (1.19)$$

where γ and Γ are related via a fluctuation-dissipation theorem.

Equation (1.17) neglects the effects of the mean field, however. If a mean field potential $V(\mathbf{r}, t)$ is included, the generalized Langevin equation can be written

$$\dot{\mathbf{v}} = -\gamma \mathbf{v} + V(\mathbf{r}, t) + L(t). \quad (1.20)$$

The fact that this mean field may be unstable has serious implications for the relaxation time of the system, which will be discussed in Chapter 4.

1.4 Evidence for Stochastic Orbits in Galactic Systems

As mentioned previously, elliptical galaxies have classical binary relaxation time-scales much longer than the age of the universe. However, many such galaxies are well fit by isothermal density distributions, as if they were fully relaxed. Thus it is apparent that the classical estimate of the relaxation time is inadequate.

A primary assumption of essentially all classical estimates of the relaxation time is the existence of an integrable mean field; stars follow a rectilinear trajectory except when they experience a two-body encounter. In three dimensions, most potentials that can be constructed are nonintegrable, leading to chaotic behavior, with integrable potentials being a set of measure zero. Unlike integrable systems, which contain only regular orbits, nonintegrable systems generally exhibit both regular and stochastic orbits, which would be expected to have drastic effects on the dynamical evolution of the system. A more detailed description of nonintegrable systems is forthcoming in chapter 2.

Nonetheless, it is still standard practice to study potentials which admit only regular orbits, the class of Stäckel potentials being an example of commonly studied models of triaxial galaxies. It could be argued that, even if the potential is not completely integrable, the fraction of irregular orbits may yet be small. However, recent evidence suggests that at least a reasonable fraction of galaxies have a significant fraction of stochastic orbits. Such evidence includes numerical studies of ‘realistic’ galactic potentials, as well as recent observational evidence obtained from the Hubble Space Telescope.

Spiral galaxies that are nearly axisymmetric may be argued to contain predominantly regular orbits. However, spiral galaxies are frequently observed that contain a large amplitude central bar. It has been shown numerically [6,7] that the existence of such bars dramatically increases the fraction of stochastic orbits. Models of barred galaxies which also include a central mass concentration were studied by Hasan and Norman [8]. They conclude that, even

though the central mass tends to dissolve the bar, orbits close to the center of the galaxy tend to become stochastic.

Significant deviations from integrability are found in models of elliptical galaxies as well. Considering the simplicity of the shape of elliptical galaxies, one may be inclined to expect that regular orbits are prevalent. This is not necessarily the case, however, and while it often would seem convenient to construct stationary configurations using nothing but regular orbits, the presence of stochastic orbits is not inconsistent with the existence of stationary quasi-equilibria.

As has long been known, typical galaxy components tend to be triaxial [9,10], and perturbations of even integrable models typically lead to a finite fraction of stochastic orbits, as is the case with Stäckel potentials perturbed by a small central singularity [11]. A substantial increase in stochasticity was also observed in numerical experiments by Udry and Pfenniger [12] with the addition of a perturbation. In these studies, triaxial galaxies included either a central mass perturbation, or a multipolar perturbation which may model the twisting of isophotes that is frequently observed.

These numerical observations are backed up by recent results from the Hubble Space Telescope (HST). Observational evidence obtained from the HST confirms and extends the results of ground-based observations of giant ellipticals, which are shown to be nonrotating, anisotropic, and moderately triaxial, and typically to have cuspy cores [13]. These cusps are due to large central mass concentrations, possibly black holes. Galaxies which are observed by the HST to have cusps have been classified as either ‘core’ galaxies or ‘power-law’ galaxies [14]. Galactic models with similar cusps have been studied by Merritt

and Fridman [15]. These authors studied strongly triaxial systems with weak cusps, for which the central density varied as $\rho \sim r^{-1}$, and strong cusps, with $\rho \sim r^{-2}$, which have density profiles similar to those of the ‘core’ and ‘power-law’ galaxies observed by HST. The results indicate that about 80 percent of the orbits with zero initial velocity are stochastic for a wide variety of energies for the strong cusp model, and, surprisingly, between 60 and 80 percent of the orbits are stochastic even for the weak cusp model, depending on the orbital energy. While these results are for systems which are more strongly triaxial than observed, they do indicate that a substantial amount of orbits in a typical realistic giant elliptical galaxy would be expected to be stochastic.

The above evidence seems to indicate that, if one wishes to consider orbits of stars in a realistic galactic potential, dynamical effects due to a finite degree of stochasticity should not be ignored. In the next chapter, we will introduce concepts from dynamical systems theory that will allow us to adequately study the dynamics of particles in such nonintegrable systems.

CHAPTER 2

CHAOS IN HAMILTONIAN DYNAMICAL SYSTEMS

The purpose of this chapter is to introduce the concepts and techniques of dynamical systems theory necessary for the understanding of chaos in Hamiltonian systems. Dynamical systems can be classified as either continuous systems, usually described as a system of differential equations, or discrete systems (maps). A system of N first-order, autonomous, ordinary differential equations can be written as $dx/dt = F[x(t)]$, where x is an N -dimensional vector. Maps can be written in vector form as $x_{n+1} = M(x_n)$, where x_n has N components. Maps are generally easier to study numerically; however, there is no universal way to convert continuous systems into discrete systems while being certain not to alter the dynamics in a qualitatively significant way. Nevertheless, under certain circumstances, Poincaré surfaces provide a useful method of reducing an N -dimensional continuous system to an $(N-1)$ -dimensional discrete system. These will be treated later in this chapter.

Let us restrict our attention to conservative, i.e. Hamiltonian, systems, for which the phase volume $\nabla_x \cdot F(x) = 0$, as required by Liouville's theorem. The N -dimensional space through which the system evolves is called *phase space*. Given an initial condition $x(0)$ (or x_0 for maps), the path through phase space along which a particle will travel is called an *orbit* or *trajectory*. These terms have slightly different meanings in some of the mathematical literature;

however, this distinction is not universal and from here on the terms will be used interchangeably.

2.1 Integrable Hamiltonian Dynamical Systems

A Hamiltonian mechanical system is given by the phase space, a symplectic structure on the phase space given by the symplectic two-form ω^2 , and a Hamiltonian function $H(p_1, \dots, p_n; q_1, \dots, q_n; t)$. By Darboux's theorem, one can choose a local coordinate system $(p_1, \dots, p_n; q_1, \dots, q_n)$ such that the symplectic two-form has the form

$$\omega^2 = \sum_{i=1}^n dp_i \wedge dq_i. \quad (2.1)$$

Then, by Stoke's lemma, a finite-dimensional, unconstrained Hamiltonian system can be characterized by Hamilton's equations of motion:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad (2.2)$$

where p_i and q_i are the phase space coordinates, position and momentum.

Consider a Hamiltonian dynamical system with N degrees of freedom; this system has a $2N$ -dimensional phase space. A Hamiltonian system is integrable if it has at least N independent constants of the motion, including the energy, for which, for any two such constants F_1 and F_2 , the Poisson bracket

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial q} \frac{\partial F_2}{\partial p} - \frac{\partial F_2}{\partial q} \frac{\partial F_1}{\partial p} \quad (1.13)$$

must vanish. The constants F_1 and F_2 are then said to be in involution. Any function which is in involution with the Hamiltonian is a constant of the motion, which can be seen using Hamilton's equations.

For a time-independent, one degree-of-freedom Hamiltonian system, since the energy is a constant of the motion, we have $H(p, q) = E$, so, up to sign ambiguities, the momentum can be written as a function of the position alone, $p = p(q, E)$. With the help of Hamilton's equations, we can then reduce to quadratures

$$t = \int_{q_0}^q \frac{dq}{(\partial H / \partial p)}. \quad (2.3)$$

Thus, the system is integrable, though we may not be able to evaluate the integral analytically.

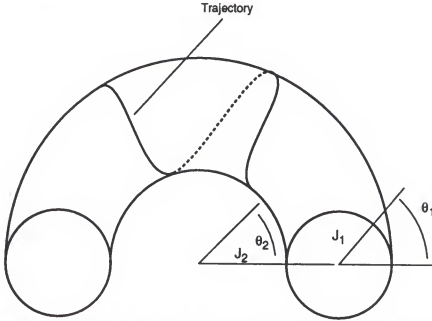


Figure 2.1 Action-angle variables. The figure shows a trajectory on the surface of a torus for a two degree-of-freedom integrable Hamiltonian.

If a Hamiltonian system is integrable, action-angle variables can be found. The action variables J_i are defined to be

$$J_i = \frac{1}{2\pi} \oint p_i dq_i, \quad (2.4)$$

and the angles θ_i are the conjugate coordinates to J_i . In this case, motion lies on an N -dimensional torus, for which the radii of the torus are determined by the actions. As a consequence of this transformation, the Hamiltonian can be written as a function of the actions only, i.e. $H = H(\mathbf{J})$; the angles are cyclic coordinates. The equations of motion are then seen to be trivial:

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial J_i} = \omega_i, \quad \frac{dJ_i}{dt} = -\frac{\partial H}{\partial \theta_i} = 0, \quad (2.5)$$

where the ω_i are the frequencies, so that the complete solution to the equations of motion is simply

$$\theta_i = \omega_i t + \phi_i. \quad (2.6)$$

For such systems the coordinates p_i and q_i can be written as a Fourier expansion of the angle variables. This is said to yield multiperiodic (or quasiperiodic) motion in p_i and q_i .

Now consider a two degree-of-freedom integrable system, for which the motion lies on a two-dimensional torus (see figure 2.1). If the two frequencies are commensurate (rational multiples of one another), then the motion is periodic and the orbit will repeat itself. However, in the generic case, the frequencies are irrational multiples of one another, and the orbit will cover the entire surface of the torus. These tori in action-angle space are called *invariant tori*.

2.2 Near-integrable Systems

Near-integrable systems are often described as perturbations of integrable systems. The Hamiltonian for such a system can be written in terms of the action-angle variables of the unperturbed motion,

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta}), \quad (2.7)$$

where H_0 is the unperturbed, integrable Hamiltonian, and ϵ is a small parameter. Even though integrability is an exceptional circumstance, many dynamical systems are known that are 'close to integrable,' for example, the three-body problem and the fast rotations of a rigid body.

In order to solve equation (2.7), one has to resort to perturbation theory. The conventional approach is to expand the Hamiltonian in a Fourier series in the parameter ϵ . A generating function can be introduced to transform to a new set of action-angle variables. The Hamiltonian for this new set of variables can be expanded in a Taylor series in ϵ . The consequence of carrying out this procedure is that the actions are now expressed, to lowest order in ϵ , with a denominator which may be small. This denominator can be expressed as a term $\mathbf{m} \cdot \boldsymbol{\omega}$, which leads to resonances when the values of $\mathbf{m}$ are such that the denominator approaches zero. As was originally shown by Poincaré [16], such perturbation expansions are expected to diverge near these internal resonances.

Kolmogorov [17], in an attempt to overcome the problem of small denominators associated with resonance zones, constructed a superconvergent perturbation theory which was applicable to nonresonant tori. For integrable systems, the phase space is completely divided into invariant tori. For a system that is close to an integrable system, Kolmogorov proved that the motion remains quasi-periodic for most initial data. Nonresonant tori, or KAM tori named after Kolmogorov, Arnol'd and Moser, are not destroyed by resonances when a sufficiently small parameter ϵ is introduced. Instead, phase space is divided into regions where invariant tori exist, and regions where resonances have destroyed the tori and motion is stochastic. The volume of phase space occupied by resonances approaches zero as ϵ approaches zero. As ϵ increases,

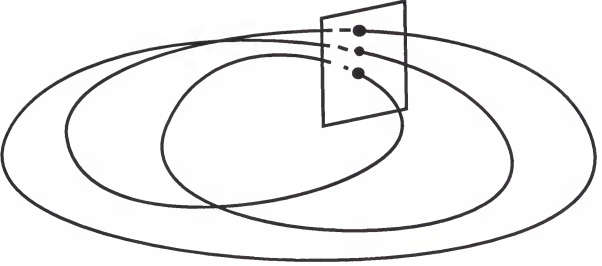


Figure 2.2 The figure shows the intersection of a periodic orbit with a Poincaré surface of section. The resulting Poincaré map consists of a period-three orbit.

the measure of the stochastic region(s) increases, as more and more KAM tori are destroyed. These ideas are more clearly seen after we introduce the concept of Poincaré surfaces of section, which is the subject of the next section.

2.3 Poincaré Maps

Poincaré had the useful idea to convert the study of continuous systems to that of discrete systems by reducing the number of dimensions of the system. The resulting system is called a Poincaré map, or surface of section. Consider an autonomous Hamiltonian system with two degrees-of-freedom, i.e., with a four-dimensional phase space. Any trajectory lies on a three-dimensional energy hypersurface $H(p_1, p_2, q_1, q_2) = H_0$, so $p_2 = p_2(p_1, q_1, q_2)$. We can then study the crossings of the trajectory across a two-dimensional surface given, say, by $q_2 = \text{const.}$, in a certain direction, say, $q_1 > 0$.

If the system is integrable, another global constant of the motion exists, $I(p_1, p_2, q_1, q_2) = \text{const.}$, and so we can write $p_1 = p_1(q_1, q_2)$. In this case successive crossings lie on regular curves on the surface of section.

If a second integral does not exist, intersections of the trajectory with the surface of section do not necessarily lie on regular curves. Consider, as an example, the Henon-Heiles potential:

$$U(x, y) = \frac{1}{2}(x^2 + y^2 + 2x^2y - \frac{1}{2}y^3). \quad (2.8)$$

Figure 2.3 is a surface of section for this potential computed for three different values of the energy [18]. For the case $E = 1/12$, it appears that all trajectories lie on smooth curves. Therefore, this potential may appear to be integrable for this value of the energy, even though it is not. As one increases the energy, however, these regular curves appear to break down. One can see that, for $E = 1/8$, the surface is partially covered by invariant tori, and part of the phase space is stochastic. For $E = 1/6$, most of the phase space is stochastic.

The construction of a Poincaré map yields several advantages for the study of differential equations, as noted by Wiggins [19].

- (1) Dimensional reduction – The construction of a Poincaré map removes at least one of the variables of the original equation. This also means that some of the information about the original system is lost.
- (2) Global dynamics – For systems with few degrees-of-freedom, computation of a Poincaré map may provide insight into the global dynamics of the system. Unfortunately, the choice of cross-section may be of importance; different surfaces of section for the same system may not necessarily have the same dynamics. However, there are situations when the Poincaré map contains all the information regarding the dynamics of the original differential equation, as, for example, when the region of phase space of interest

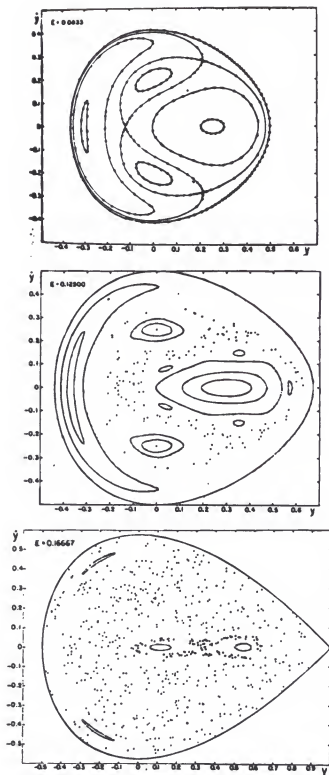


Figure 2.3 Poincaré surface of section for the Henon-Heiles potential

(a) $E = 1/12$, (b) $E = 1/8$, (c) $E = 1/6$

contains only periodic orbits. (3) Conceptual clarity – A fixed point in the surface of section corresponds to a periodic orbit of the original differential equation. It can be shown that if a fixed point of the Poincaré map is asymptotically stable, then the corresponding periodic orbit is also asymptotically stable. It is easier to study the stability of the fixed point, which is simply characterized in terms of the eigenvalues of the map linearized about the fixed point, than the stability of the periodic orbit in the original differential equation. Results concerning the dynamics of the differential equation may be more clearly stated for the Poincaré map.

Moreover, it is apparent that maps are much easier to study numerically than differential equations. Two-dimensional maps that are dynamically similar to the surfaces of section of nonintegrable Hamiltonians can be easily constructed, as, e.g., the figures associated with the Henon-Heiles potential. Thus, studying the dynamics of such maps should yield useful information regarding the dynamics of differential equations. For simplicity, it would be preferable to learn as much as possible about two-dimensional Hamiltonian mappings before tackling the more difficult task of studying Hamiltonian differential equations, with the understanding that such a mapping should be qualitatively similar to a Poincaré map.

2.4 Invariant Manifolds and Homoclinic Points

Invariant manifolds, including stable and unstable manifolds, are important analytical tools for the study of dynamical systems. A set S is said to be *invariant* if, under the action of the vector field or map, a point within S remains within S for all time. As far as we are concerned, a manifold can be considered to be a set which *locally* has the structure of Euclidean space. In

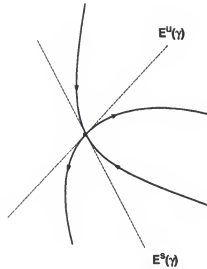


Figure 2.4 The figure shows one-dimensional stable and unstable manifolds of a fixed point, γ . The stable and unstable manifolds of this point, W^s and W^u , are tangent to the linear subspaces E^s and E^u .

nonlinear settings, it is sufficient to consider a manifold to be an m -dimensional surface embedded in R^n .

Note that when we call something a stable or unstable manifold, it has to be *of* something. Most often, these manifolds are considered to be of fixed points. However, invariant manifolds of more general sets, e.g., typical stochastic trajectories in Hamiltonian systems, may also be of importance.

For fixed points, the stable, unstable and center manifolds span the subspaces determined by their eigenvalues, sufficiently close to that fixed point; positive eigenvalues correspond to unstable manifolds, negative to stable manifolds, and zero to center manifolds. For fixed points, the matrix of first partial derivatives is constant. For more general points, however, this is not true and calculation of the manifolds are better done numerically (by linearizing, as for fixed points). Note that, as illustrated in figure 2.4, the nonlinear manifolds are

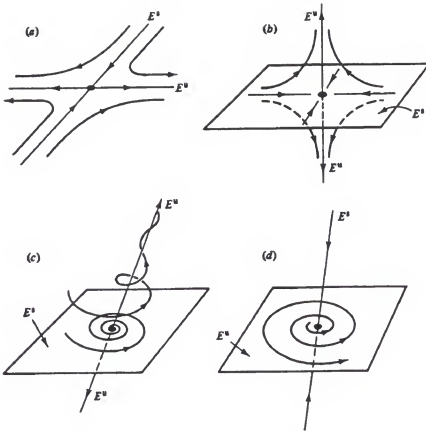


Figure 2.5 The figure shows one- and two-dimensional stable and unstable linear subspaces of a fixed point. Nearby trajectories tend to diverge from the fixed point in the unstable directions and converge towards the fixed point in the stable directions.

tangent to the associated linear manifolds at the location of the fixed point. Figure 2.5 [20] shows several one- and two-dimensional stable and unstable subspaces and the behavior of nearby trajectories.

Consider the stable and unstable manifolds of a hyperbolic periodic point γ . A hyperbolic point is a point for which the manifolds intersect transversely, that is, non-tangentially. It is sufficient to consider the periodic point to be a fixed point, simply by applying any argument for f to f^p , where f is the given map, and f^p is the p^{th} iterate of the map. A point that is within the intersection of the manifolds is said to be *homoclinic* to γ . The existence of

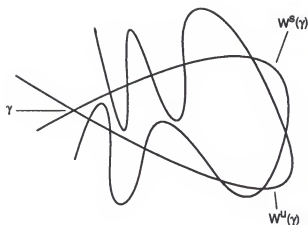


Figure 2.6 The figure shows a homoclinic tangle caused by the infinite number of intersections of the stable and unstable manifolds of a fixed point.

such points is an important ‘trigger’ for chaotic behavior. It can also be shown that the existence of a single such point implies infinitely many such points. This is called the *homoclinic tangle*, and is shown in figure 2.6. Moreover, transverse intersections of the stable and unstable manifolds are a necessary condition for structural stability. Difficulty occurs when the intersections are not transverse. When they are tangent, we obtain a *homoclinic tangency*, and the assumption of hyperbolicity is lost. Regions where such a tangency occurs are then nonhyperbolic. Since physical systems are, in general, nonhyperbolic, the behavior of the stable and unstable manifolds is of vital importance in understanding the structural stability of dynamical systems.

2.5 Liapunov Exponents and the Definition of Chaos

Liapunov exponents provide an important tool in the understanding of the stochastic regions of phase space. Nearby orbits in stochastic regions diverge

from each other exponentially. This is called sensitive dependence on initial conditions, and is the fundamental phenomenon associated with chaotic behavior. Liapunov characteristic exponents are a primary quantity in determining the existence and rate of this divergence.

Consider two trajectories with initial conditions $\mathbf{x}(0)$ and $\mathbf{x}(0) + \delta\mathbf{x}(0)$, respectively. Exponential sensitivity to initial conditions would imply that, for a long enough time t , $\delta\mathbf{x}(t) \sim \exp(\lambda t)\delta\mathbf{x}(0)$, for some positive λ . In a regular region where exponential sensitivity does not occur, one would expect that $\lambda \rightarrow 0$ as $t \rightarrow \infty$.

The Liapunov exponents can be explicitly defined as follows [21,22]. Given an initial condition $\mathbf{x}(0)$ and an initial separation vector $\delta\mathbf{x}(0)$, we can define the mean exponential rate of divergence of the trajectories to be

$$\lambda(\mathbf{x}(0), \delta\mathbf{x}) = \lim_{\substack{t \rightarrow \infty \\ d(0) \rightarrow 0}} \left(\frac{1}{t} \right) \ln \frac{d(\mathbf{x}(0), t)}{d(\mathbf{x}(0), 0)}, \quad (2.9)$$

where $d(\mathbf{x}(0), t)$ is simply the norm of $\delta\mathbf{x}(t)$, usually chosen to be the Euclidean norm, though it is of course independent of the choice of norm.

One can then find an orthogonal basis $\{\hat{e}_i\}$ of $\delta\mathbf{x}$ such that

$$\lambda_i(\mathbf{x}(0)) = \lambda(\mathbf{x}(0), \hat{e}_i), \quad (2.10)$$

where the λ_i are the Lyapunov exponents. Once the largest Liapunov exponent is found, the others can be found using an orthogonalization procedure, such as that of Gram-Schmidt. The largest Liapunov exponents for numerous initial conditions in the Henon-Heiles potential were calculated by Benettin et al. [21]. Several of these exponents are re-calculated here, and shown in figure 2.7.

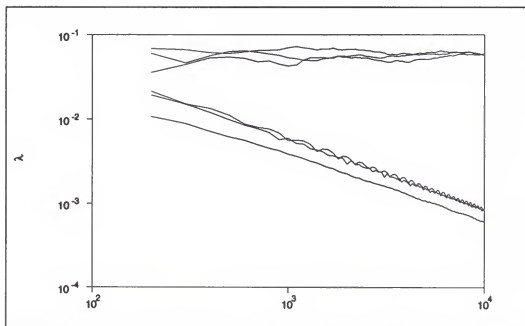


Figure 2.7 The figure shows the largest Liapunov exponent for six sets of initial conditions in the Henon-Heiles potential, with $E = 0.125$. Three of these are within the stochastic region and seem to converge to the same value, while the other three are within the regular region and tend towards zero.

For Hamiltonian systems, Liouville's theorem implies that the number of positive Liapunov exponents equals the number of negative ones. Hence two-dimensional autonomous Hamiltonian systems have only one positive Liapunov exponent.

Despite the fact that an exact definition of chaos is not universally agreed upon, the existence of positive Liapunov exponents is critical to any reasonable definition. Devaney's topological definition of chaos is frequently cited [23]. According to this definition, there are three properties all chaotic systems must possess: (1) a dense set of periodic orbits; (2) topological transitivity, the property that an orbit beginning in one arbitrarily small neighborhood will eventually pass through any other; and (3) sensitive dependence on initial conditions. It has actually recently been shown that, generically, conditions

(1) and (2) imply condition (3) [24] . It would be possible, however, to find a system which satisfies condition (3) while not obeying (1) and (2), although such a system would be pathological.

Reichl [25] defines a chaotic system to be a system with positive Kolmogorov entropy, the sum of the positive Liapunov exponents. Such systems are called K-systems. A more restricted set of systems are C-systems, which are defined to be systems that have a decomposable linear tangent space, i.e., these are systems that are everywhere hyperbolic. Such systems are not very physical in the sense that this suggests that the Hamiltonian is discontinuous. These hyperbolic systems may be considered the most chaotic, in the sense that, for any choice of initial conditions, one has the maximum exponential sensitivity. In any case, calculating a positive Liapunov exponent is a good indication that chaos is present in a given system.

2.6 Dissipative Systems

Most often, chaos is considered in the realm of dissipative systems, which are characterized by the presence of attractors in the phase space. Chaotic dissipative systems have strange attractors, on which trajectories have sensitive dependence on initial conditions. Conservative dynamical systems, i.e., Hamiltonian systems, have no such attractors because trajectories in the phase space are volume preserving – attractors are characterized by a reduction of dimension. Nevertheless, regions of the phase space in *nonintegrable* Hamiltonian systems are often ‘chaotic’ in this sense that they have sensitive dependence on initial conditions. Such systems are often called stochastic, as their phase space is divided into ‘regular’ and ‘stochastic’ regions.

Dissipative dynamical systems can be studied in much the same way as Hamiltonian systems are. The standard methods of determining Poincaré surfaces, stable and unstable manifolds, and Liapunov exponents still apply. Many systems of physical interest are best modelled by dissipative dynamical systems. However, nature may be Hamiltonian at its most fundamental level, and the complex dynamics that arise in nonintegrable Hamiltonian systems make these worthy of study in themselves.

CHAPTER 3 CHAOS IN THE GRAVITATIONAL N-BODY PROBLEM

In general, the problem of N bodies interacting via Newton's law of gravitation is nonintegrable for $N > 2$. The full $6N$ -dimensional problem, in fact, has ten integrals of motion, related to the initial conditions of the center of mass (3), the total linear momentum (3), the total angular momentum (3), and the energy (1). The full N -body system has been shown numerically to be chaotic in the sense that it has positive Lyapunov exponents. Consider evolving an N -body system in its $6N$ -dimensional phase space, and evolving a second such system which is very nearly identical, i.e. the system's location in phase space is very close to that of the first system. Numerical studies indicate that, for $N > 2$, these two trajectories will diverge from each other exponentially, which is an indication of the presence of chaos.

We are of course primarily interested in the case when $N \gg 2$, for the case of galactic dynamics. For the study of such systems, numerous numerical codes exist, a standard one being Aarseth's, found, e.g., in Appendix 4.B of Galactic Dynamics [2]. The primary purpose of such codes is to integrate the equations of motion directly,

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial U}{\partial \mathbf{q}_i}, \quad (3.1)$$

where

$$U = - \sum_{1 \leq i \leq j \leq N} \frac{G m_i m_j}{\| \mathbf{q}_i - \mathbf{q}_j \|}. \quad (3.2)$$

Computationally, equation (3.2) leads to problems, since the denominator blows up when $\mathbf{q}_i$ is close to $\mathbf{q}_j$. The standard method of dealing with this problem is to insert a ‘softening parameter’ into the equation, so that the softened potential reads

$$U = - \sum_{1 \leq i \leq j \leq N} \frac{Gm_i m_j}{\sqrt{r^2 + \epsilon^2}}, \quad (3.3)$$

where ϵ is the softening parameter, and $r = \|\mathbf{q}_i - \mathbf{q}_j\|$.

The softening parameter can also be interpreted as ‘turning off’ the effects of close encounters of nearby particles in the N -body simulation. One can thus determine the effect that this close encounter ‘noise’ has on the instability of the gravitational N -body problem, compared to the effect of the bulk mean field of the system. In this chapter, this effect will be studied by performing N -body simulations to determine the time-scale of instability for various values of the softening parameter.

3.1 Method

This section describes the method by which the time-scale of instability for each simulation is determined. The simulations were run using an N -body code developed by Haywood Smith, which is similar to the basic code of Aarseth.

For each case, the particles were distributed randomly with uniform probability within a sphere of radius $R_0 = 1$, with velocities chosen similarly, but scaled such that the initial virial ratio of the system, $-2E_k/E_p = 1$. The particles then interacted via forces derived from equation (3.3). A perturbed system was then generated in an identical fashion, except that each particle was displaced by a distance $\delta r = 2 \times 10^{-8}$ from the unperturbed particles in a randomly chosen direction.

These simulations were run for systems comprised of $N = 360$ particles, 300 of which had a mass $m = 1$ (light particles) and the remaining 60 of which had $m = 5$ (heavy particles). In addition, some simulations were run for systems comprising of $N = 996$ particles, with the same ratio of light/heavy particles. Three different runs were performed for each value of ϵ , and the results from these runs were averaged to extract mean properties.

For each pair (perturbed and unperturbed) of simulations, the perturbation of each particle's position, $\delta \mathbf{r}_i(t)$, was determined, as well as the perturbation in velocity, $\delta \mathbf{v}_i(t)$. These data were then analyzed to determine two types of quantities.

First, the time taken for each perturbation $\delta \mathbf{r}_i(t)$ to grow to a value e times the initial $\delta \mathbf{r}_i(0) = 2 \times 10^{-8}$ was determined. These times were then analyzed to extract mean e -folding times $t_{*,L}$ and $t_{*,H}$ for the light and heavy particles, as well as the standard deviations $\sigma_{*,L}$ and $\sigma_{*,H}$. To ensure that the qualitative results were not dependent on the choice of e as an amplification factor, an identical analysis was also effected for the 1.2-folding times.

The total $3N$ -dimensional configuration and velocity space perturbations were computed as well. These are defined to be

$$\Delta r \equiv \frac{1}{2} \ln \left(\sum_i |\delta \mathbf{r}_i|^2 \right) \quad \text{and} \quad \Delta v \equiv \frac{1}{2} \ln \left(\sum_i |\delta \mathbf{v}_i|^2 \right). \quad (3.4)$$

The corresponding quantities Δr_L , Δv_L , Δr_H and Δv_H , associated individually with the light and heavy particles, were also computed. Growth time scales $t_{\Lambda,L}$ and $t_{\Lambda,H}$ were then extracted, derived from the quantities Δr_L and Δr_H via

$$\Delta r(t) = \Delta r(0) + (t/t_{\Lambda}). \quad (3.5)$$

Characteristic growth times for the velocity perturbations were determined as well, which agreed with the time scales extracted from the configuration space perturbations but, owing to the larger irregularities, exhibited larger statistical uncertainties.

3.2 Results

Table 3.1

Average e -folding times and Standard Deviations ($N = 360$)

ϵ	$t_{*,L}/t_{cr}$	$t_{*,H}/t_{cr}$	$\sigma_{*,L}/t_{cr}$	$\sigma_{*,H}/t_{cr}$
10^{-8}	0.47 ± 0.01	0.49 ± 0.04	0.21 ± 0.01	0.21 ± 0.01
0.005	0.50 ± 0.01	0.50 ± 0.05	0.23 ± 0.01	0.21 ± 0.01
0.01	0.54 ± 0.01	0.55 ± 0.03	0.23 ± 0.01	0.22 ± 0.01
0.02	0.66 ± 0.02	0.66 ± 0.06	0.26 ± 0.01	0.24 ± 0.02
0.05	1.09 ± 0.02	1.12 ± 0.05	0.41 ± 0.02	0.41 ± 0.05
0.08	1.71 ± 0.03	1.79 ± 0.08	0.56 ± 0.03	0.54 ± 0.02
0.08	2.65 ± 0.03	2.49 ± 0.35	0.90 ± 0.02	0.95 ± 0.07
0.08	3.58 ± 0.25	3.79 ± 0.30	1.02 ± 0.16	1.00 ± 0.22

The mean e -folding times for both light and heavy particles, and the associated dispersions, are tabulated in Table 3.1 for the different values of ϵ . It can be seen immediately that the e -folding times $t_{*,L}$ and $t_{*,H}$ for fixed ϵ are in fact equal to within statistical uncertainties. In terms of the first two moments, there are no discernable differences between light and heavy particles. This similarity extends also to the full distributions of e -folding times, which

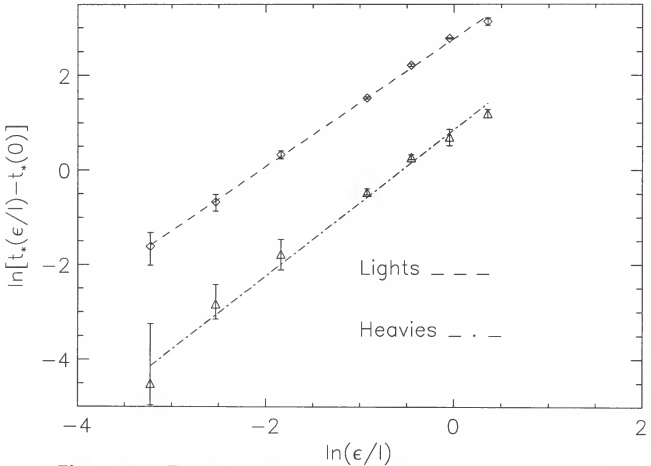


Figure 3.1 The mean e-folding times for light and heavy particles as a function of ϵ/l , the ratio of the softening parameter to the interparticle spacing.

were binned and examined graphically. There is no evidence whatsoever that the distributions for the light and heavy particles are not identical.

It is also observed that, for $\epsilon \leq 0.18$, the values of the time scales $t_{*,L}$ and $t_{*,H}$ are smooth, monotonically increasing functions of ϵ . In fact, these values are consistent with a power-law growth with respect to the softening parameter, if it is expressed in units of the typical interparticle spacing, $l \equiv \Gamma(4/3)N^{-1/3}R_0 \approx 0.890N^{-1/3}$, computed following, e.g., Chandrasekhar (1941). These data are easily fit to

$$t_*(\epsilon/l) = t_*(0) + A_*(\epsilon/l)^p, \quad (3.6)$$

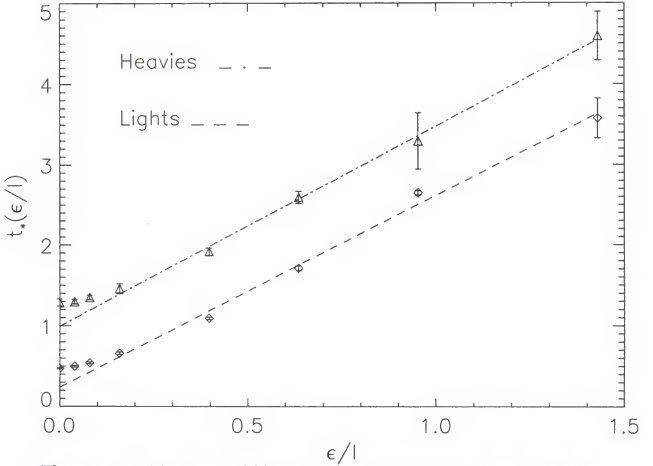


Figure 3.2 The mean e-folding times for light and heavy particles as a function of ϵ/l , plotted on a linear scale.

with best fit values $p_L = 1.35 \pm 0.08$ and $A_L = 2.15 \pm 0.03 t_{cr}$ for the light particles and $p_H = 1.55 \pm 0.23$ and $A_H = 2.35 \pm 0.16 t_{cr}$ for the heavy particles. If the point $\epsilon = 0.18$ be excluded from the fit, the values of the exponents are increased to $p_L = 1.38$ and $p_H = 1.60$. The goodness of fit is illustrated in figure 3.1. Because there are only $1/5$ as many heavy particles as light particles, the uncertainties in $t_{*,H}$ are quite large, so large that p_H and p_L are in agreement within statistical uncertainties. Indeed, if one excludes the value $\epsilon = 0.005$, the best fit value of p_H reduces to 1.37.

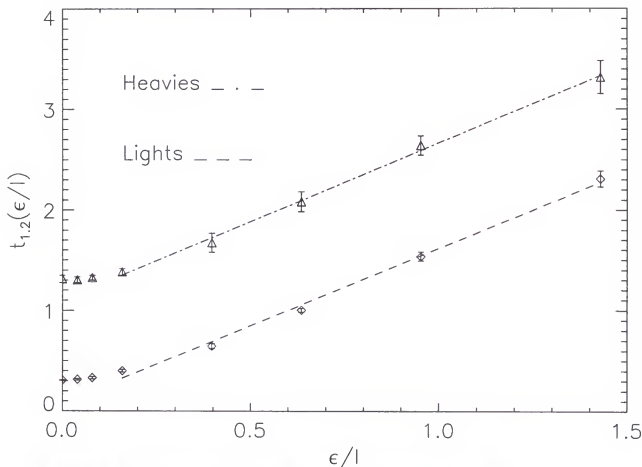


Figure 3.3 The mean 1.2-folding times for light and heavy particles as a function of ϵ/l .

The values for the ϵ -folding times can also be fit by a simple linear relationship if values $\epsilon < 0.02$ are excluded. The best fit values for

$$t_*(\epsilon/l) = t_*(0) + B_*(\epsilon/l) \quad (3.7)$$

are then $t_{*,L}(0) = 0.24t_{cr}$ and $B_{*,L} = 2.38t_{cr}$ for the light particles and $t_{*,H}(0) = 0.19t_{cr}$ and $B_{*,H} = 2.48t_{cr}$ for the heavy particles. The goodness of fit is illustrated in figure 3.2.

The 1.2-folding times $t_{1,2,L}$ and $t_{1,2,H}$, plotted in Figure 3.3, are also well fit by a simple power law growth. In this case, the best fits are $p_{1,2,L} = 1.50 \pm 0.12$

and $A_{1.2,L} = 1.54 \pm 0.05 t_{cr}$ for the light particles and $p_{1.2,H} = 1.50 \pm 0.18$ and $A_{1.2,H} = 1.54 \pm 0.08 t_{cr}$ for the heavies. The data are thus consistent with the interpretation that the ratios $t_{1.2,L}/t_{*,L}$ and $t_{1.2,H}/t_{*,H}$ are independent of ϵ , although there is a hint that the 1.2-folding times may grow somewhat more rapidly. For both the light and heavy particles, the best fit value for the ratio is $t_{1.2}/t_* = 0.61 \pm 0.03$.

Table 3.2Growth Times ($N = 360$)

ϵ	$t_{\Lambda,L}/t_{cr}$	$t_{\Lambda,H}/t_{cr}$
10^{-8}	0.10 ± 0.01	0.09 ± 0.01
0.005	0.15 ± 0.01	0.16 ± 0.01
0.01	0.18 ± 0.02	0.18 ± 0.01
0.02	0.28 ± 0.02	0.27 ± 0.03
0.05	0.52 ± 0.02	0.52 ± 0.07
0.08	0.80 ± 0.03	0.82 ± 0.04
0.08	1.13 ± 0.06	1.27 ± 0.03
0.08	1.17 ± 0.21	1.34 ± 0.25

The mean growth times t_{Λ} for the total perturbations Δr_L and Δr_H of the light and heavy particles are tabulated in Table 3.2. Again, no difference can be seen between the results for the light and heavy particles, to within statistical uncertainties.

These growth times are smooth, monotonically increasing functions of ϵ as well. As illustrated in figure 3.4, they are well fit by a power law

$$t_{\Lambda}(\epsilon/l) = t_{\Lambda}(0) + A_{\Lambda}(\epsilon/l)^p. \quad (3.8)$$

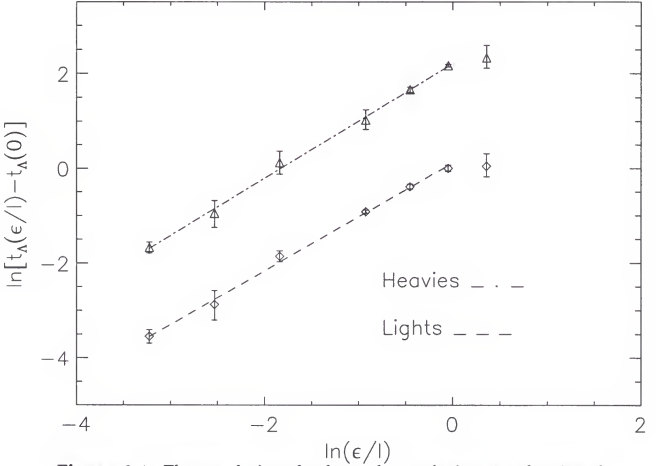


Figure 3.4 The growth times for the total perturbations as a function of ϵ/l .

The best fit values here are $p_{\Lambda,L} = 1.13 \pm 0.05$ and $A_{\Lambda,L} = 1.12 \pm 0.04 t_{cr}$ for the light particles and $p_{\Lambda,H} = 1.12 \pm 0.06$ and $A_{\Lambda,H} = 1.23 \pm 0.05 t_{cr}$ for the heavy particles.

Several simulations were run for the case $N = 996$ as well. We had assumed previously that the instability time scales depended only on the value of the ratio ϵ/l . The mean interparticle spacings for the systems with $N = 360$ and $N = 996$ differ by a factor of $(360/996)^{1/3} \approx 0.712$. If our assumption is

Table 3.3Average ϵ -folding times and Standard Deviations ($N = 360$)

ϵ	N	$t_{*,L}/t_{cr}$	$t_{*,H}/t_{cr}$	$\sigma_{*,L}/t_{cr}$	$\sigma_{*,H}/t_{cr}$
10^{-8}	360	0.47 ± 0.01	0.49 ± 0.04	0.21 ± 0.01	0.21 ± 0.01
10^{-8}	996	0.45 ± 0.01	0.46 ± 0.02	0.19 ± 0.01	0.19 ± 0.01
0.0200	360	0.66 ± 0.02	0.66 ± 0.06	0.26 ± 0.01	0.24 ± 0.02
0.0143	996	0.67 ± 0.01	0.67 ± 0.03	0.26 ± 0.02	0.25 ± 0.02
$t_{1.2,L}/t_{cr}$	$t_{1.2,H}/t_{cr}$	$\sigma_{*,L}/t_{cr}$	$\sigma_{*,H}/t_{cr}$	$t_{\Lambda,L}/t_{cr}$	$t_{\Lambda,H}/t_{cr}$
0.31 ± 0.01	0.31 ± 0.03	0.19 ± 0.01	0.18 ± 0.01	0.10 ± 0.01	0.09 ± 0.01
0.30 ± 0.01	0.30 ± 0.01	0.18 ± 0.01	0.18 ± 0.01	0.12 ± 0.02	0.12 ± 0.02
0.40 ± 0.01	0.38 ± 0.03	0.24 ± 0.01	0.22 ± 0.02	0.28 ± 0.03	0.27 ± 0.05
0.40 ± 0.01	0.41 ± 0.01	0.23 ± 0.01	0.22 ± 0.01	0.28 ± 0.04	0.30 ± 0.03

correct, the time scales should be identical for these two systems by decreasing the softening parameter by a factor of 0.712 for the case of 996 particles.

The results for 996 particles are tabulated and compared to the results for 360 particles in table 3.3, for two values of the softening parameter. In these cases, the predicted scaling is confirmed, and it can also be seen visually that the shapes of the distributions are identical. Thus, the effect of softening seems to be independent of the number of particles, which implies that close encounters remain important even for large N .

3.3 Representative Particle Trajectories

The form of the individual particle perturbations $|\delta\mathbf{r}|$ and $|\delta\mathbf{v}|$ and, particularly, the question of the degree to which different particles behave similarly, was studied by extracting and then examining individually the perturbations

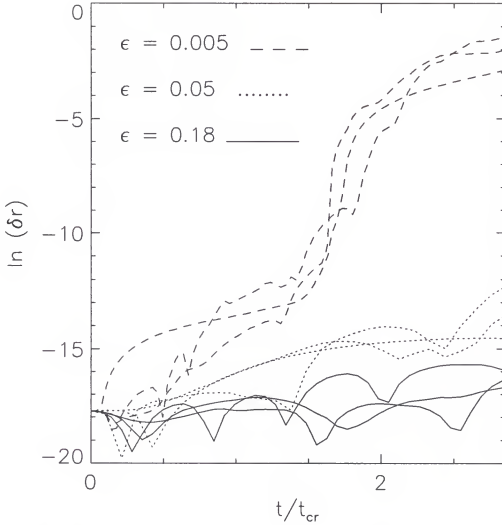


Figure 3.5 The separation of typical trajectories $|\delta r(t)|$ for three individual particles each with $\epsilon = 0.005, 0.05$, and 0.18 .

of 5% each of the light and heavy particles in several different simulations. It was immediately evident that there are no significant differences between the trajectories of the light and heavy particles, and, for this reason, in the following no attempt has been made to distinguish between the light and heavy particles. This analysis of individual particles was effected for three different values of epsilon, $\epsilon = 0.005, 0.05$, and 0.18 . Three runs each were analyzed for

$\epsilon = 0.005$ and 0.18 , but only one run was analyzed for the intermediate value $\epsilon = 0.05$. The behavior of three representative particles for each of these values of ϵ is illustrated in Figure 3.5.

It is evident that, regardless of the value of ϵ , there is a roughly exponential growth for the individual particle perturbations, interwoven, however, with significant "bumps." This overall qualitative behavior can be interpreted as representing a sum of two effects, namely (1) a smooth growth, which can be attributed to the bulk mean field, and (2) sudden, random kicks which can be attributed to shorter range interactions between particularly proximate particles. Significantly, however, there is an initial transient period, lasting about $0.5t_{cr}$ to $1t_{cr}$, during which the perturbation does *not* exhibit an overall rapid exponential growth. In fact, the perturbation tends initially to either decrease somewhat or to only grow relatively slowly. A more rapid growth does not appear to set in until it is triggered by some initial shorter range encounter.

For all values of ϵ , the individual velocity perturbations $|\delta\mathbf{v}|$ are considerably more jagged than the configuration space perturbations, a phenomenon long observed for the evolution of the total $3N$ -dimensional perturbations $\Delta\mathbf{r}$ and $\Delta\mathbf{v}$ [26]. However, a comparison of the individual $|\delta\mathbf{r}|$'s and $|\delta\mathbf{v}|$'s yields qualitative differences for different values of ϵ . For the smallest softening parameter, namely $\epsilon = 0.005$, the kicks in the velocity perturbation tend to occur at the same times as the kicks in the configuration space perturbation. However, for $\epsilon = 0.18$, the kicks in velocity tend to be completely out of phase with the kicks in position. The delay in the velocity perturbation of the kicks due to short range encounters becomes significant when there is a large amount of

softening. For $\epsilon = 0.18$, it was also observed that the “bumps” in the configuration and velocity space perturbations sometimes appear oscillatory: some 15% of the randomly chosen particles can be said to display this sort of behavior.

Since, for such a large softening, the perturbation never grows very large (the typical $|\delta\mathbf{r}| < 10^{-6}$ even at the end of the simulations), it seemed possible that these “oscillations” might be associated with perturbations of some binary system. This hypothesis was in fact studied in some detail, but the results were inconclusive. It was found that, although a single other particle *did* often tend to remain in the neighborhood of the “oscillating” particle during the entire run, the times at which this particle approached the first could not be simply correlated with the kicks in the perturbation of the original particle. This problem clearly warrants future attention.

Overall, the typical growth of individual perturbations is at least roughly fit by a simple exponential form $|\delta\mathbf{r}(t)| \sim |\delta\mathbf{r}(0)| \exp(t/t_\lambda)$. However, as for the total $3N$ -dimensional perturbation, the growth time t_λ clearly depends on the value of ϵ . When probed over a time of approximately $2.8t_{cr}$, the mean growth times t_λ are comparable to, but slightly longer than, the growth times t_Λ for the full $3N$ -dimensional perturbations. Indeed, one finds that, for $\epsilon = 0.005$, $t_\lambda = 0.17 \pm 0.01 t_{cr}$, for $\epsilon = 0.05$ that $t_\lambda = 0.59 t_{cr}$, and, for $\epsilon = 0.18$ that $t_\lambda = 1.41 \pm 0.37 t_{cr}$. These values are to be compared with the time scales $t_{\Lambda,L} = 0.15 \pm 0.01 t_{cr}$, $0.52 \pm 0.02 t_{cr}$, and $1.17 \pm 0.21 t_{cr}$ and the corresponding $t_{\Lambda,H}$ recorded in Table 2. The fact that these individual time scales are slightly longer reflects the fact that the full $3N$ -dimensional perturbation is dominated by those particles whose perturbation $|\delta\mathbf{r}|$ is growing more quickly than average. When probed only over the initial $0.7t_{cr}$, the corresponding time

scales t_λ are increased by roughly a factor of two. In this case, one finds respectively the values $t_\lambda = 0.23 \pm 0.04 t_{cr}$, $1.01 t_{cr}$, and $2.40 \pm 0.09 t_{cr}$. This difference reflects the fact that, initially, the perturbation either decreases or, at best, increases only very slowly.

A related question concerns the degree to which the perturbations of different particles in fact grow at roughly the same rate. For $\epsilon = 0.005$, the growth rates tend to agree to within about 10% on a time scale of $2.8 t_{cr}$ and to within $\sim 50\%$ over $0.7 t_{cr}$, but, over the first $0.35 t_{cr}$, only at the 175% level. For $\epsilon = 0.18$, the agreement is at the 10% level over $5.6 t_{cr}$, at the 50% level over $2.8 t_{cr}$, and at the 350% level over $0.7 t_{cr}$. Over the initial $0.35 t_{cr}$, the agreement actually improves to $\sim 75\%$.

This agreement during the initial transient period for $\epsilon = 0.18$ is related to the fact that, for a majority of the particles, the perturbation actually decreases initially. This decrease is in fact observed for every value of ϵ , but it becomes more significant for the larger values: the initial decrease is almost always larger for $\epsilon = 0.18$, and the frequency of its occurrence is greater. Moreover, the perturbation reaches its minimum earlier for the lowest values of ϵ . For $\epsilon = 0.005$, the minimum almost always occurs before $0.35 t_{cr}$ whereas, for $\epsilon = 0.18$, this usually occurs between 0.35 and $0.7 t_{cr}$. For $\epsilon = 0.005$, the initial growth is negative for some 60% of the particles, for the 0.05 case, $\sim 70\%$ of the particles, and, for $\epsilon = 0.18$, $\sim 80\%$ of the particles.

For those particles for which the perturbation does not initially decrease, there is almost invariably only a smooth, very slow initial growth, corresponding typically to a time scale $\sim 3 - 5 t_{cr}$. There are almost no instances in which the perturbation begins to grow very quickly at the outset. Rapid growth in

the perturbation appears to begin rather abruptly at some later time, presumably after an initial kick associated with some close encounter. This fact reinforces the picture that both close encounters and the bulk mean field play an important role in the instability. It is evident that, without close encounters, the perturbation will only grow very slowly, but it is also clear that the mean field plays a role in amplifying the effects of the close encounters.

CHAPTER 4

COLLISIONAL RELAXATION IN A NONINTEGRABLE POTENTIAL

It was seen in the previous chapter that the instability of the N -body gravitational problem depends on *both* the mean field and ‘noise’ due to close encounters, independent of the size of the system. close encounters remain important even when N is large, and decrease the time scale for instability. This would imply that for galactic systems, the collisionless Boltzmann equation, as discussed in section 1.2, may not be applicable when determining the time scale for instability.

Alternatively, one could formulate a Langevin equation, which incorporates the effects of close encounter noise directly. The standard paradigm is to model these close encounters essentially as a Brownian process, while ignoring the effects of the bulk mean field [27,28]. This decoupling of the mean field and the fluctuations due to close encounters implies that the collisional relaxation, which was discussed in section 1.1, is much greater than the age of the universe for galactic systems.

This conclusion is a direct consequence of the assumption that fluctuations due to close encounters can be completely decoupled from the bulk mean field. This may indeed be true for systems in which the trajectories are stable. Non-integrable systems, however, are characterized by the existence of exponentially unstable trajectories. Thus, the standard calculation of the relaxation time of

a particle in a mean field is reexamined here, assuming the mean field trajectories are exponentially unstable, and characterized by a positive Liapunov exponent.

4.1 Relaxation in an Inverted Harmonic Oscillator

Here we consider collisional relaxation in an exactly soluble toy model, the inverted harmonic oscillator, in which all trajectories are exponentially unstable. The calculation is similar to that of Chandrasekhar [27], and can be extracted from this via an analytic continuation $\omega^2 \rightarrow -\omega^2$.

Consider the generalized one-dimensional Langevin equation

$$\frac{d^2 x}{dt^2} + \eta \frac{dx}{dt} - \omega^2 x + F(t) = 0, \quad (4.1)$$

where $F(t)$ is the stochastic force and η is the associated dynamical friction. As a simplifying idealization, assume that $F(t)$ is delta-correlated white noise with zero mean, so that F and η are related via a fluctuation-dissipation theorem

$$\langle F(t_1)F(t_2) \rangle = \Theta \eta \delta(t_1 - t_2), \quad (4.2)$$

where Θ defines a characteristic ‘temperature’, i.e. a typical mean squared velocity. Also assume that η , the coefficient of dynamical friction, is independent of position and velocity.

It is straightforward to right down the solutions to an ordinary differential equation for the damped inverted oscillator, with $F(t)$ being considered a driving term, ignoring for the moment the fact that $F(t)$ is a stochastic force. The solutions to the homogeneous equation can be expressed as

$$x = a_1 e^{\mu_1 t} + a_2 e^{\mu_2 t} \quad (4.3)$$

where μ_1 and μ_2 are the roots of the characteristic equation

$$\mu^2 + \eta\mu - \omega^2 = 0. \quad (4.4)$$

To determine a particular solution, it is convenient to use the method of variation of parameters, as discussed in, for example, [29]. Thus a_1 and a_2 are replaced by functions of time and determined by the equations

$$\dot{a}_1 e^{\mu_1 t} + \dot{a}_2 e^{\mu_2 t} = 0 \quad (4.5)$$

and

$$\mu_1 \dot{a}_1 e^{\mu_1 t} + \mu_2 \dot{a}_2 e^{\mu_2 t} = F(t). \quad (4.6)$$

The equations (4.5) and (4.6) can be used to obtain integral solutions for $a_1(t)$ and $a_2(t)$, which can be inserted into equation (4.3) to obtain

$$x - x_0 e^{-\eta t/2} \cosh \frac{\beta t}{2} - \frac{x_0 \eta + 2v_0}{\beta} e^{-\eta t/2} \sinh \frac{\beta t}{2} = \int_0^t d\xi F(\xi) \Psi(\xi) \quad (4.7)$$

and

$$v - v_0 e^{-\eta t/2} \cosh \frac{\beta t}{2} + \frac{v_0 \eta - 2\omega^2 x_0}{\beta} e^{-\eta t/2} \sinh \frac{\beta t}{2} = \int_0^t d\xi F(\xi) \Phi(\xi), \quad (4.8)$$

where $\beta^2 = \eta^2 + 4\omega^2$. The right-hand sides of equations (4.7) and (4.8) involve the functions

$$\Psi(\xi) = \frac{2}{\beta} e^{-\eta(t-\xi)/2} \sinh[\beta(t-\xi)/2] \quad (4.9)$$

and

$$\Phi = d\Psi/dt. \quad (4.10)$$

At this point, it is important to consider that $F(t)$ is a stochastic force, and hence the equations (4.7) and (4.8) can only be solved in the sense of specifying

a probability distribution function which can be used to extract average quantities for the variables of interest. Given the form of the left-hand sides of equations (4.7) and (4.8), the distributions functions $W(x, t; x_0, v_0)$, $W(v, t; x_0, v_0)$, and $W(x, v, t; x_0, v_0)$ can be written in terms of the integrals

$$\int_0^t d\xi \Psi^2(\xi); \quad \int_0^t d\xi \Phi^2(\xi) \quad \text{and} \quad \int_0^t d\xi \Psi(\xi) \Phi(\xi) \quad (4.11)$$

as proven, for example, by Chandrasekhar [27].

More straightforwardly, the averages of the moments $\langle x^p v^q \rangle$ can be computed directly from equations (4.7) and (4.8). Since the stochastic force has zero mean, averaging the equations (4.7) and (4.8) immediately gives

$$x = x_0 e^{-\eta t/2} \cosh \frac{\beta t}{2} - \frac{x_0 \eta + 2v_0}{\beta} e^{-\eta t/2} \sinh \frac{\beta t}{2} \quad (4.12)$$

and

$$v = v_0 e^{-\eta t/2} \cosh \frac{\beta t}{2} + \frac{v_0 \eta - 2\omega^2 x_0}{\beta} e^{-\eta t/2} \sinh \frac{\beta t}{2}, \quad (4.13)$$

while the second moments can be obtained using the integrals (4.11).

In an astronomical context, the instability would be expected to grow on the time scale of a fiducial crossing time, i.e. the dynamical time t_D , so ω would be roughly comparable to t_D^{-1} . The coefficient of dynamical friction, which is related to the noise via a fluctuation-dissipation theorem, would be determined by the time scale set by close encounters, so that, by definition, $\eta \equiv t_R^{-1}$, where t_R is the classical binary relaxation time. Recall from chapter 1 that t_R is much greater than the age of the universe, so that $t_D \ll t_R$.

It is also useful to neglect transients and focus on the behavior that occurs after several crossing times, as well as neglecting effects which proceed on the much longer time t_R . Thus, we wish to assume that $\omega^{-1} \ll t \ll \eta^{-1}$.

In this approximation, the noise and friction are weak and $\beta \approx 2\omega$. Since we are neglecting transients, $\sinh \omega t \approx e^{\omega t}/2$. Thus, the solutions (4.12) and (4.13) reduce to

$$\langle x \rangle = \frac{1}{2} \exp[(\omega - \eta/2)t] \left(x_0 + \frac{v_0}{\omega} + \frac{\eta}{2\omega} x_0 \right) \quad (4.14)$$

and

$$\langle v \rangle = \frac{1}{2} \exp[(\omega - \eta/2)t] \left(v_0 + \omega x_0 - \frac{\eta}{2\omega} v_0 \right). \quad (4.15)$$

This result reduces to the deterministic result $F(t) \rightarrow 0$ when the friction coefficient $\eta \ll 2\omega$, so that $2\omega - \eta \approx 2\omega$.

In the weak noise and weak friction limit $\beta \approx 2\omega$, the integrals (4.11) simplify as well and can be written as

$$\int_0^t d\xi \Psi^2(\xi) \approx \frac{1}{8\omega^3} \exp[(2\omega - \eta)t], \quad (4.16)$$

$$\int_0^t d\xi \Phi^2(\xi) \approx \frac{1}{8\omega} \exp[(2\omega - \eta)t], \quad (4.17)$$

and

$$\int_0^t d\xi \Psi(\xi) \Phi(\xi) \approx \frac{1}{8\omega^2} \exp[(2\omega - \eta)t]. \quad (4.18)$$

With the help of the fluctuation-dissipation theorem (equation (4.2)), we can then write the second moments as

$$\langle x^2 \rangle = \langle x \rangle^2 + \frac{\Theta\eta}{8\omega^3} \exp[(2\omega - \eta)t], \quad (4.19)$$

$$\langle v^2 \rangle = \langle v \rangle^2 + \frac{\Theta\eta}{8\omega} \exp[(2\omega - \eta)t], \quad (4.20)$$

and

$$\langle xv \rangle = \langle x \rangle \langle v \rangle + \frac{\Theta\eta}{8\omega^2} \exp[(2\omega - \eta)t]. \quad (4.21)$$

These moment equations can now be used to determine approximations for the growth rates for the mean squared changes in position and velocity $\langle |\delta x|^2 \rangle$

and $\langle |\delta v|^2 \rangle$, as well as $\langle \delta x \delta v \rangle$. To lowest order in η , $\langle |\delta x|^2 \rangle \approx \langle x^2 \rangle - \langle x \rangle^2$ so that, for $2\omega - \eta \approx 2\omega$,

$$\langle |\delta x|^2 \rangle = \frac{\Theta \eta}{8\omega^3} e^{2\omega t}, \quad (4.22)$$

and similarly

$$\langle |\delta y|^2 \rangle = \frac{\Theta \eta}{8\omega} e^{2\omega t} \quad (4.23)$$

and

$$\langle \delta x \delta y \rangle = \frac{\Theta \eta}{8\omega^2} e^{2\omega t}. \quad (4.24)$$

Recall from chapter 1 that the relaxation time was defined in terms of changes in the mean square velocity, the classical result being

$$\frac{\langle |\delta v|^2 \rangle}{\langle v^2 \rangle} \sim \frac{\ln N}{N} \frac{t}{t_D}. \quad (4.25)$$

This can be contrasted to the current estimate obtained from equation (4.23), using the facts that the ‘temperature’ Θ corresponds to a mean square velocity $\langle v^2 \rangle$, $\omega \sim t_D^{-1}$, $\eta \equiv t_R^{-1}$, and, from the virial theorem, $\frac{t_R}{t_D} \sim \frac{N}{\ln N}$ (equation 1.6). The conclusion is that collisional effects in an inverted harmonic oscillator, for which the mean field orbits are unstable, induce changes in the mean square velocity of trajectories which are estimated to be

$$\frac{\langle |\delta v|^2 \rangle}{\langle v^2 \rangle} \sim \frac{\ln N}{N} \exp(2t/t_D). \quad (4.26)$$

Similar conclusions are obtained for changes in the other moments as well.

It is apparent that, even if N is very large, the exponential factor in equation (4.26) implies that the time scale for appreciable changes in velocity will be relatively short. Thus, the effective relaxation time for trajectories in a nonintegrable potential can be orders of magnitude smaller than the classical binary relaxation time, for which stable mean field orbits are implied.

The above conclusion is not necessary valid for changes in energy, however. The mean change in particle energy can be estimated from the above moment equations to yield, for $\langle \delta E \rangle = \frac{1}{2} \langle v^2 - \omega^2 x^2 \rangle - \frac{1}{2} (v_{mf}^2 - \omega^2 x_{mf}^2)$,

$$\langle \delta E \rangle = - \left(\frac{\eta}{8\omega} \right) (v_0^2 - \omega^2 x_0^2) e^{2\omega t}. \quad (4.27)$$

For this toy model, perturbation in particle energy induced by the noise and friction will indeed grow exponentially. However, this is a consequence of the fact that, for this model, the unperturbed positions and velocities diverge for late times, even though the unperturbed particle energy is in fact conserved. for a generic potential, this is not true and the natural time-scale associated with changes in particle energy is, in fact, the classical relaxation time, $t_R \equiv \eta^{-1}$.

While the above conclusions were arrived at for the case of an exactly soluble toy model, the same basic idea can be exploited for any generic potential with unstable mean field orbits. This is the idea explored in the next section.

4.2 Relaxation in a Generic Nonintegrable Potential

Consider a particle moving in a potential $U(r)$ so that it follows a smooth, deterministic trajectory $r^a(t)$ which obeys

$$\frac{d^2 r^a}{dt^2} + \frac{\partial U(r)}{\partial r^a} = 0 \quad (a = 1, 2, 3). \quad (4.28)$$

The background potential $U(r)$ may be nonintegrable and have some exponentially unstable orbits. If we now include a fluctuating force and its associated friction, we obtain a generalized Langevin-type equation of the form

$$\frac{d^2 R^a}{dt^2} + \frac{\partial U(R)}{\partial R^a} = -\eta \frac{dR^a}{dt} + F^a, \quad (4.29)$$

solutions of which represent noisy trajectories $R^a(t)$.

The primary interest is in the difference between these trajectories, $\delta r^a = R^a - r^a$. Treating δr^a as a perturbation and linearizing, we obtain

$$\frac{d^2 \delta r^a}{dt^2} + \eta \frac{\delta r^a}{dt} + \frac{\partial^2 U(r)}{\partial r^b \partial r^a} \delta r^b = F^a - \eta v^a, \quad (4.30)$$

where $v^a = dr^a/dt$.

Since we are assuming the orbit r^a is exponentially unstable, at least *one* of the eigenvalues of the matrix $\frac{\partial^2 U(r)}{\partial r^b \partial r^a}$ must be negative, and hence of the form $-\omega^2$ where $\omega^2 > 0$. It is reasonable to assume that the stochastic force F^a acts randomly in all directions, so that it will eventually probe the direction associated with ω_A^2 . In any event, if the orbit is exponentially unstable in all directions, so that for each value of a the eigenvalues are of the form $-\omega_a^2$, considering only the smallest value of ω_a^2 will only underestimate the importance of the instability. Thus, it is reasonable to replace $\frac{\partial^2 U(r)}{\partial r^b \partial r^a}$ by $-\omega^2 \delta^{ab}$, where ω^2 can be considered the smallest value of ω_a^2 .

The value ω can be interpreted as a Liapunov exponent for the mean field trajectory, which must be positive for exponentially unstable directions. While we are interpreting ω to be the smallest Liapunov exponent to be safe, it should be apparent that the direction associated with the largest Liapunov exponent should quickly dominate the instability, and hence should be the physically relevant direction. Indeed, numerical simulations indicate that the largest Liapunov exponent generally provides a reasonable characterization of the instability, as shown in, e.g. [30].

The linearized equation can now be written

$$\frac{d^2 \delta r^a}{dt^2} + \eta \frac{\delta r^a}{dt} - \omega^2 \delta r^a = F^a - \eta v^a. \quad (4.31)$$

which is an inhomogeneous stochastic differential equation for δr^a , with the quantity $F^a - \eta v^a$ acting as the source. This equation is very similar to equation (4.1) for the toy model. Thus it is straightforward to write down solutions to equation (4.31) in terms of the functions Ψ and Φ given in equations (4.9) and (4.10). The resulting equations are

$$\delta r^a = \int_0^t d\xi [F^a(\xi) - \eta v^a(\xi)] \Psi(\xi) \quad (4.32)$$

and

$$\frac{d\delta r^a}{dt} = \delta v^a = \int_0^t d\xi [F^a(\xi) - \eta v^a(\xi)] \Phi(\xi). \quad (4.33)$$

In order to evaluate the moments of this equation, it is again convenient to assume delta-correlated Gaussian white noise with zero mean, so that the three-dimensional fluctuation-dissipation theorem becomes

$$\langle \mathbf{F}(t_1) \cdot \mathbf{F}(t_2) \rangle = \Theta \eta \delta_D(t_2 - t_1). \quad (4.34)$$

Therefore equation (4.32) can be used to determine the second moment

$$\langle |\delta \mathbf{r}|^2 \rangle = \left\langle \int_0^t d\xi \int_0^t d\zeta \Psi(\xi) \Psi(\zeta) [F^a(\xi) - \eta v^a(\xi)] [F_a(\zeta) - \eta v_a(\zeta)] \right\rangle \quad (4.35)$$

Since the friction and the noise are uncorrelated and the noise is assumed to have zero mean, this can be written

$$\langle |\delta \mathbf{r}|^2 \rangle = \Theta \eta \int_0^t d\xi \Psi^2(\xi) + \eta^2 \left| \int_0^t d\xi \mathbf{v}(\xi) \Psi(\xi) \right|^2. \quad (4.36)$$

Similar results can of course be obtained for $\langle |\delta \mathbf{v}|^2 \rangle$ and $\langle \delta \mathbf{r} \cdot \delta \mathbf{v} \rangle$.

It is appropriate to evaluate these quantities in the late time, weak noise limit considered in the last section. In this case, the second term in equation (4.36) can be neglected for typical values of the velocity, since it is of order η^2 .

Since this term is positive definite, by doing so we are only underestimating the growth rate. The remaining integral is given by equation (4.16) and hence the result is

$$\langle |\delta \mathbf{r}|^2 \rangle = \frac{\Theta \eta}{8\omega^3} e^{2\omega t}. \quad (4.37)$$

Similarly,

$$\langle |\delta \mathbf{v}|^2 \rangle = \frac{\Theta \eta}{8\omega} e^{2\omega t}. \quad (4.37)$$

and

$$\langle \delta \mathbf{r} \cdot \delta \mathbf{v} \rangle = \frac{\Theta \eta}{8\omega^3} e^{2\omega t}. \quad (4.37)$$

We thus recover the same results as for our model in section 2, namely, that the relaxation time associated with changes in the mean square of the phase space variables may be orders of magnitude shorter than the classical result.

CHAPTER 5

NONHYPERBOLIC SHADOWING IN MAPPINGS

The fact that chaotic dynamical systems exhibit sensitive dependence on initial conditions implies that the presence of even weak noise may radically alter the details of individual orbits. We have seen in previous chapters (chapter 3 and chapter 4) how noise effects the time scale of the instability in a nonintegrable potential. But how does the noise affect the details of the individual orbits? More importantly, even though the individual orbits are drastically altered by the presence of noise, are the macroscopic properties of the system altered as well? In other words, do the noisy orbits behave the same as the deterministic orbits, in a statistical sense, or is the behavior of the orbits in a noisy chaotic system qualitatively changed?

The answer to these questions is of extreme importance: if the behavior of the noisy orbits is qualitatively similar to the orbits in the deterministic system, we might expect that the statistics of the orbits obtained for both systems should be identical. And since we know that for a chaotic system the character of the individual orbits are essentially useless, due to the exponential divergence of trajectories, the addition of noise would not hinder any understanding of the orbits in this regard. For such systems, why bother analysing the noisy system at all, since one would expect to obtain identical results for the corresponding deterministic system.

The above scenario is not always true however, and it is important to understand the qualitative change that noise has on typical orbits in a non-integrable system. In the next few chapters, I will attempt to answer these questions using shadowing theorems.

5.1 Shadowing as Noise Reduction

In many cases, noise is an unwanted and annoying artifact in physics, and it is often hoped that one can minimize its effects. Undesirable noise may be introduced into physical experiments via errors associated with the measurement process, and into computer experiments via roundoff and truncation errors. Noise due to measurement errors can generally be considered *observational noise*, sometimes called additive noise, in which noise introduced at one instant does not affect the orbit at the next instant. In this case, given a dynamical system $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$, a noisy orbit is related to the deterministic orbit by simply adding a noise term η , $\mathbf{p}_n = \mathbf{x}_n + \eta_n$.

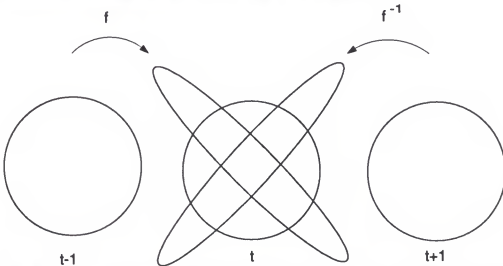


Figure 5.1 Noisy measurements made at three different times. When these measurements are transported to the same point in time, they distort in accordance with the local derivatives of the dynamical system.

In this case, there is a straightforward method to reduce unwanted noise called *nonlinear smoothing*. Consider making noisy measurements at times $t-1$, t , and $t+1$, as described by Farmer and Sidorowich [31], and shown in figure 5.1. The measurement at time $t-1$ is evolved forward in time to time t , and expands along the unstable manifold while contracting along the stable manifold. Evolving the measurement at time $t+1$ backwards to time t causes the data to expand along the stable manifold and contract along the unstable manifold. The intersection of these three data sets at time t yield a more refined measurement. The amount that the transported data set distorts depends on the rate of contraction or expansion along the stable or unstable manifold, taking into account the fact that, for Hamiltonian systems, the area of the data region must remain the same. This scheme fails when these manifolds do not lie in distinct directions, a difficulty which will be discussed more later.

By contrast, *Dynamical noise* introduced at one instant alters the dynamics of the orbit in the future, and in a chaotic system causes a divergence from the purely deterministic orbit. Computer errors are often the cause of dynamical noise. Roundoff errors are due to the finite floating point arithmetic which occurs whenever one does a computer simulation involving irrational numbers, whereas truncation errors occur whenever attempting to integrate a differential equation numerically, and depend on the details of the integration scheme used.

Even so, the effect of computer errors on the behavior of particle orbits is often ignored by the galactic dynamicist doing numerical simulations. It is frequently assumed that, even if the details of the noisy orbits differ drastically from the deterministic orbits, the properties of the orbits are the same, in some

statistical sense. Statistical tests seem to indicate that this is indeed true for computer simulations of N-body gravitational systems [32,33].

Moreover, in some cases the neglect of computer errors in a simulation can perhaps be justified by the possibility that they may actually mimic the random perturbations that influence real, physical systems. In the case of galactic dynamics, these perturbations may be assumed to be due to interactions amongst the various components of the galaxy, for example, close encounters of nearby stars. While it may be true that the computer noise will not exactly reproduce the physical noise, both types of noise are dynamical, and the exact form of the noise may not really matter. If this be true, one should not necessarily be concerned with the use of noise reduction methods to remove the impact of noise on a simulation. Instead, the concern should be whether the amplitude of the noise has any physical relevance. The existence of computer errors, or any dynamical noise for that matter, should not necessarily be considered destructive in a computer simulation, but may actually be useful for modelling a given physical system.

Nevertheless, the galactic dynamicist should still be concerned as to the qualitative effect a given amount of noise will have on the properties of orbits in a chaotic dynamical system. A potentially useful technique for determining the qualitative effect of noise on a dynamical system may be to study the *shadowability* of the system. Shadowable systems possess the property that, in a certain sense, a deterministic orbit remains close to a noisy orbit. What exactly is meant by the term shadowability will be made clear in the next sections.

While shadowing was originally motivated by the need for a noise reduction scheme for chaotic systems, I will be interpreting the shadowing property instead as a measure of the change in qualitative behavior of trajectories in the noisy system, as opposed to the deterministic system. In other words, whereas the traditional discussion of shadowing considered the deterministic orbit to be the true orbit of the system of interest while the noisy orbit is considered the undesired orbit, I will be considering the noisy orbit to be the orbit of interest, with the goal of determining how well a low-dimensional, deterministic system can model the more relevant, noisy physical system. In what follows, I will be using the traditional definitions for the orbits, so readers familiar with shadowing arguments of other authors will be able to follow the discussion.

5.2 Hyperbolic Shadowing

Hyperbolic systems, as mentioned in Chapter Two, are systems for which the stable and unstable manifolds always intersect transversely. Such systems have been proven to be shadowable for all time by Anosov [34] and Bowen [35].

Theorem. (*Anosov-Bowen, 1967*) *For hyperbolic systems, given a shadow distance $\varepsilon > 0$, there exists a $\delta > 0$ such that any δ -pseudo-orbit can be ε -shadowed by a true orbit for all time.*

This theorem concerns the orbits of a hyperbolic map $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$. A *true orbit* is defined to be the set of points $\{\mathbf{x}_n\}_{n=0}^N$, such that $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ for $n = 0, \dots, N-1$. Likewise, a δ -pseudo-orbit is the set of points $\{\mathbf{y}_n\}_{n=0}^N$ such that $|\mathbf{y}_{n+1} - f(\mathbf{y}_n)| < \delta$ for $n = 0, \dots, N-1$. In other words, the δ -pseudo-orbit is the noisy orbit with noise amplitude δ ; it remains within δ of

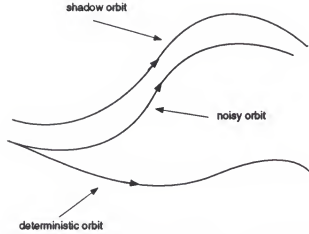


Figure 5.2 Deterministic and noisy orbits of a dynamical system. The lowest curve is the purely deterministic orbit for a given initial condition. The middle curve is the noisy orbit with the same initial condition. The upper curve is the shadow orbit that remains within a distance ϵ of the noisy orbit.

the true orbit which the system would have followed at that time step without any noise.

Now, the ϵ -shadow orbit is a true orbit of $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ if $|\mathbf{y}_n - \mathbf{x}_n| < \epsilon$ for $n = 0, \dots, N - 1$. The distance ϵ , the maximum distance between the shadow orbit and the pseudo-orbit, will be called the *shadow distance*. The hyperbolic shadowing theorem states that a deterministic orbit and a noisy orbit remain within a distance ϵ as long as they are separated by no more than δ initially. Thus, the shadow orbit and the noisy orbit do not, in general, have the same initial conditions, and, as such, the initial conditions of an orbit in a hyperbolic dynamical system need not be known exactly to determine the qualitative behavior of the orbit. This is shown in figure 5.2.

It has been shown by Anosov [36] that hyperbolic systems are structurally stable, i.e. a perturbed hyperbolic system remains hyperbolic. For such systems, one would not expect any change in the statistical quantities calculated

for the original hyperbolic system and the perturbed system. It is thus apparent that the addition of noise in a hyperbolic system will not induce any qualitative change in the behavior of orbits in the system, even though the noisy orbits are drastically different from the deterministic ones. This is true irrespective of the form of the noise: observational or dynamical, white or colored, gaussian or not.

The shadowing theorem obviously holds for any system with observational noise, hyperbolic or not. In this case, since the presence of the noise does not alter the dynamics, the shadow distance is then merely equal to the noise amplitude. The power in the shadowing theorem lies in its predictions for the case of dynamical noise.

It is important to note that, in the standard interpretation, the true orbit in the Anosov-Bowen theorem is only a true orbit of the purely deterministic dynamical system. However, if one wishes to consider the noise to be of some physical origin, then the δ -pseudo-orbit should more appropriately be considered a true orbit of the noisy dynamical system. Given this new interpretation, the hyperbolic shadowing theorem states that a true, noisy orbit with noise amplitude δ would remain within ε of some orbit of the purely deterministic system.

5.3 Shadowing of Nonhyperbolic Systems

However, most problems of physical interest are *not* hyperbolic, and the Anosov-Bowen theorem is not applicable to the majority of systems that the physicist or galactic dynamicist would like to represent. Two representative systems which are considered paradigms of chaotic systems are the Hénon

map and the standard map, neither of which is hyperbolic. Other commonly studied systems of physical interest which are nonhyperbolic are the Hénon-Heiles system, often used as a simple galactic model, and the complex Ginsburg-Landau equation, which has important applications in condensed matter physics. Hence, a more general shadowing theorem is necessary to determine the shadowability of nonhyperbolic systems.

The difficulty in shadowing nonhyperbolic systems lies in the existence of homoclinic tangencies and near-tangencies. Homoclinic tangencies occur when the angle between the stable and unstable manifolds of a point along a trajectory approaches zero. Referring back to figure 1, this means that the ovals representing the noisy measurements lie along the same axis. How close to a homoclinic tangency a trajectory is able to get depends strongly on the noise amplitude. Referring to figure 3, as the angle θ_n approaches zero, the trajectory approaches a homoclinic tangency. If θ_n is small enough so that the noise allows the trajectory to "jump" over a stable manifold, it will then diverge from the deterministic trajectory. The system has approached a near-tangency, and the trajectory is no longer shadowable.

Thus, the trajectories in nonhyperbolic systems cannot be shadowed for all time, and there is no a priori guarantee that the noisy and deterministic trajectories behave in a similar fashion. However, trajectories *can* be shadowed as long as they remain in regions which are "hyperbolic enough," that is, as long as they remain far enough away from homoclinic near-tangencies. The length of time an orbit remains in such a sufficiently hyperbolic region will be called the *shadowing time*. The 'shadowing time' may be defined as the length

of time N such that a shadow orbit exists given a noisy orbit for all times $n = 0, 1, \dots, N$, but such an orbit does not exist for $n = 0, 1, \dots, N + 1$.

Several algorithms have been implemented that calculate shadowing times for one- and two-dimensional maps. In general, these procedures rely on a refinement procedure which will be described in the next section. These algorithms have limitations that are avoided in a more powerful algorithm due to Sauer and Yorke [37]. They derive a nonhyperbolic shadowing theorem that suggests an algorithm which can be used to calculate how long an orbit can be shadowed before approaching a near-tangency, for a general map f of arbitrary dimension.

The Sauer and Yorke algorithm relies on determining how small the angle between the stable and unstable manifolds, θ_n , is allowed to get before a near-tangency is encountered. This requires information regarding the linearized map; specifically, the bounds on the growth of f_n in the direction S_n and of f_n^{-1} in the direction U_n . For this, we need to determine r_n such that $|J_n y| \leq r_n |y|$ for vectors y in S_n , and t_n such that $|J_n^{-1} y| \leq t_n |y|$ for vectors y in U_n .

Theorem. (*Sauer-Yorke, 1991*) Assume $\delta < \frac{1}{20m^2}$, where m is the dimension of the system, and let B be a bound on the first and second partial derivatives of the map f and its inverse f^{-1} . Define $C_0 = 0$; $C_n = \csc \theta + r_n C_{n-1}$ for $n > 0$ and $D_N = 0$; $D_n = \csc \theta_n + t_n D_{n+1}$ for $n < N$. If

$$\max\{C_n, D_n\} \leq \frac{1}{m^{5/2} B^2 \sqrt{\delta}} \quad (5.1)$$

for all $n = 0, \dots, N$, then there exists an orbit $\{y_n\}$ of f such that $|x_n - y_n| < \sqrt{\delta}$ for $n = 0, \dots, N$.

As before, δ is the noise amplitude, x_n is the noisy orbit, and y_n is the shadow orbit. The time step N is considered to be the shadowing time, while

the location of the orbit at this time step is the location of the near-tangency. The quantities C_n and D_n are obtained through recursion relations involving the evolution of the stable and unstable manifolds along the orbit at each time step n . At the time-step for which either C_n or D_n gets larger than the bound determined by the theorem, a homoclinic near-tangency has been encountered and the system is no longer shadowable.

The nonhyperbolic shadowing theorem contains the hyperbolic shadowing theorem of Anosov and Bowen as a specific case; for hyperbolic systems, the quantities C_n and D_n remain bounded from above, and never violate the bound given in the above theorem. Hence for hyperbolic systems the shadowing time N is infinite.

The Sauer and Yorke theorem also quantifies the shadow distance ϵ . While in the Anosov-Bowen theorem the shadow distance was only known to exist but never explicitly specified, in the new theorem it is seen that this shadow distance $\epsilon = \sqrt{\delta}$, which is the maximum distance between the noisy and deterministic orbits as long as the orbit is not integrated beyond $n = N$.

5.4 The Nonhyperbolic Shadowing Algorithm

The primary method for finding shadow orbits includes the *refinement procedure* of Hammel, Yorke, and Grebogi [38]. Given a noisy and deterministic trajectory, the idea is to refine the noisy orbit by applying a recursion relation to the components of the noisy orbit until the noise can no longer be reduced. The refinement procedure as given in [38] and used in [39] and [40] is as follows.

Let $\mathbf{e}_{n+1}$ represent the one-step error for the noisy orbit, and Δ_n represent the distance between the noisy orbit and a deterministic orbit at time step n :

$$\mathbf{e}_{n+1} = \mathbf{y}_{n+1} - f(\mathbf{y}_n), \quad (5.2)$$

$$\Delta_n = \mathbf{x}_n - \mathbf{y}_n. \quad (5.3)$$

We can then combine these equations to write

$$\Delta_{n+1} = f(\mathbf{x}_n) - f(\mathbf{y}_n) - \mathbf{e}_{n+1}. \quad (5.4)$$

We now assume that Δ_n , being the shadow distance for a given time step, is small, so we can expand $f(\mathbf{x}_n)$ in a Taylor series and linearize to obtain

$$f(\mathbf{x}_n) \simeq f(\mathbf{y}_n) + J_n \Delta_n, \quad (5.5)$$

where J_n is the Jacobian of the appropriate map; if the dynamical system is a system of differential equations, then J_n is the Jacobian of the map determined by the integration routine. So equation (5.4) becomes

$$\Delta_{n+1} = J_n \Delta_n - \mathbf{e}_{n+1}. \quad (5.6)$$

We wish to find $\{\Delta_n\}_{n=0}^N$ in terms of the coordinates defining the stable and unstable manifold. So we will rewrite Δ_n and $\mathbf{e}_n$ in terms of the variables $\mathbf{u}_n$ and $\mathbf{s}_n$. These are the unit basis vectors which lie along the unstable and stable directions; they can be constructed using the Gram-Schmidt orthogonalization procedure. So

$$\Delta_n = \alpha_n \mathbf{u}_n + \beta_n \mathbf{s}_n, \quad (5.7)$$

and

$$\mathbf{e}_n = \eta_n \mathbf{u}_n + \zeta_n \mathbf{s}_n. \quad (5.8)$$

The above equations can then be substituted into equation (5.6), and after some algebraic manipulation, we obtain the recursion relations

$$\alpha_n = (\alpha_{n+1} + \eta_{n+1})/|J_n \mathbf{u}_n|, \quad \alpha_N = 0, \quad (5.9)$$

and

$$\beta_{n+1} = \beta_n |J_n \mathbf{s}_n| - \zeta_{n+1}, \quad \beta_0 = 0. \quad (5.10)$$

Computing these coefficients, one is able to find a trajectory that is less noisy than the original one. Near a homoclinic tangency, however, no improvement over the noise may be possible.

Some difficulties are inherent in this procedure, however. First of all, the refinement computations must be carried out at a higher accuracy than the noise level. Thus, if it is desired to shadow trajectories with double-precision level computer noise, then computations using 96-bit arithmetic are required. This is obviously impractical. An added complication is the dependence of the shadow distance on the one-step error. It thus may be necessary to consider many different realizations of the noise before a reasonable estimate of the shadow distance is obtained.

The algorithm obtained from the new shadowing theorem does not have these difficulties. First of all, computations need only be carried out to the same precision as the noise. Also, the refinement process is included in the theorem itself, where the bounds on the noise level are used instead of the actual one-step error. It is hence not necessary to know the exact form of the noise in order to calculate the shadowing time. Of course, this simplification in the numerical process of calculating shadowing results leads to the limitation

of only obtaining bounds on the shadowing time, and thus the results obtained must be interpreted somewhat differently.

Recall from chapter two that invariant manifolds are defined anywhere within a hyperbolic set. Given an initial condition of a differential equation, the stable and unstable manifolds can be calculated anywhere along the trajectory, until a homoclinic tangency is reached. The stable and unstable directions, assuming they are each one-dimensional, are determined by the equations

$$\mathbf{u}_{n+1} = J_n \mathbf{u}_n / |J_n \mathbf{u}_n|, \quad (5.11)$$

$$\mathbf{s}_{n+1} = J_n \mathbf{s}_n / |J_n \mathbf{s}_n|. \quad (5.12)$$

The unstable manifold at each point can be calculated by choosing a random vector and iterating equation (5.11) forwards in time, while the stable manifold at that point is calculated by choosing a random vector and iterating equation (5.12) backwards. If the manifolds are more than one-dimensional, than Gram-Schmidt orthogonalization can be used.

CHAPTER 6

RESULTS OF SHADOWING NONHYPERBOLIC SYSTEMS

The nonhyperbolic shadowing theorem that was proven by Sauer and Yorke was used as a basis for numerical experiments involving the shadowability of various nonhyperbolic systems. These numerical experiments involved utilizing the algorithm outlined in the previous chapter, primarily to calculate the shadowing time and the location in the system where shadowing breaks down. This chapter will discuss these shadowing results for a number of nonhyperbolic systems.

6.1 The Model Systems

Shadowing results were obtained for a variety of systems including both maps and differential equations. The maps studied are the Hénon map and the standard map; the differential equations include truncations of the Toda lattice potential and the anisotropic Kepler potential. While the potential systems are more relevant physically, in fact, both the Toda lattice truncations and the anisotropic Kepler problem have applications in galactic dynamics and condensed matter physics, the maps are easier to analyze and computationally less time-consuming. Maps are also relevant for study because, as mentioned in chapter 2, they can be considered reasonable approximations to Poincaré maps of differential equations.

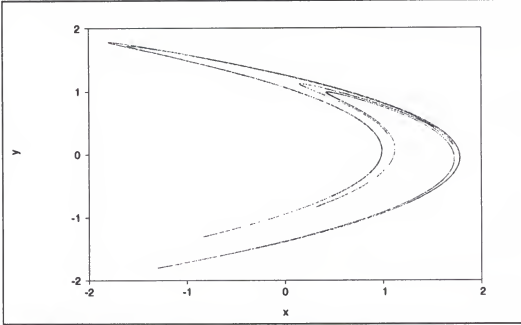


Figure 6.1 The Hénon map, with $a=1.4$ and $b=0.3$.

As a first example, consider the Hénon map [41]:

$$\begin{aligned} x_{n+1} &= by_n + a - x_n^2 \\ y_{n+1} &= x_n. \end{aligned} \tag{6.1}$$

This is a dissipative map as long as b is chosen to be less than one, independent of the value of a . In this study, the parameters a and b were chosen to be 1.4 and 0.3, respectively, so that iteration of the map yields the familiar strange attractor shown in figure 6.1.

As an example of a conservative map, the shadowing algorithm was applied to the standard map, which, in action-angle variables, has the form

$$J_{n+1} = J_n + (K/2\pi) \sin 2\pi \theta_n \pmod{1}, \tag{6.2}$$

$$\theta_{n+1} = \theta_n + J_{n+1} \pmod{1}. \tag{6.3}$$

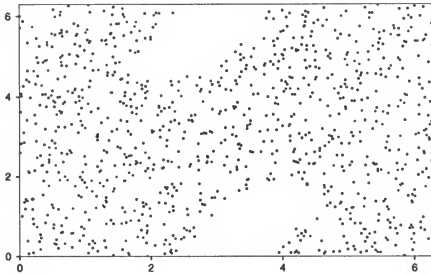


Figure 6.2 A single stochastic trajectory in the standard map, for $K = 3$.

This map is nonhyperbolic for all positive values of the stochasticity parameter K [39].

The standard map is a good example of a generic nonhyperbolic area-preserving map, conveniently displaying the important qualities present for such systems. For a typical value of K , the phase plane is divided into regions consisting entirely of regular orbits and regions consisting entirely of orbits that wander stochastically around the portion of the phase plane accessible to them. These regions are separated from each other by KAM tori. When K is small, the majority of the phase space is covered by regular orbits, with isolated regions of stochasticity associated with the resonance zones. As K increases, the measure of the stochastic region of phase space also increases, as KAM tori break down into cantori, and the cantori themselves begin to disappear. At a critical value of K , appropriately called the *critical value*, the last KAM torus that separates regions of stochasticity disappears; above this value the

standard map exhibits *global stochasticity*. A typical stochastic orbit is shown in figure 6.2 for $K = 3$, which is above the critical value. The regions where the orbit does not visit are occupied by regular orbits, and separated from this stochastic orbit by KAM tori.

The first differential equation studied was the Hamiltonian system generated by the Hénon-Heiles potential. The Hénon-Heiles potential was studied by Hénon and Heiles [42] in the 1960's as a simple model of a galaxy which includes stochastic orbits. This is a two degree-of-freedom system with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \quad (6.4)$$

with the potential given by

$$V(x, y) = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3. \quad (2.8)$$

An interesting example of an integrable Hamiltonian system is the Toda lattice [43]. The periodic three-particle Toda lattice is a chain of three particles connected by non-linear springs, and is given by the Hamiltonian

$$H = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \exp(q_1 - q_2) + \exp(q_2 - q_3) + \exp(q_3 - q_1) - 3. \quad (6.5)$$

Recall that, in order to be integrable, this system needs to have two constants of motion besides the energy. These constants have been proven to exist and have been found by Flaschka [44] and Hénon [45]. Flaschka's method involved the use of *Lax pairs* [46]. A Lax pair is a pair of matrices, say L and M , whose elements depend on a parameter t such that $\frac{dL}{dt} = LM - ML$. The constants of the motion are then the coefficients in the characteristic polynomial $\det(L - \lambda I)$, which do not vary with t .

The three-particle Toda lattice can be reduced to a two-degree of freedom system via a transformation

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{24}[\exp(2\sqrt{3}x + 2y) + \exp(-2\sqrt{3}x + 2y) + \exp(-4y)] - \frac{1}{8}. \quad (6.6)$$

Expanding the exponential terms up to sixth order, we obtain

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2 + y^2) + x^2y - \frac{1}{3}y^3 + \frac{1}{2}x^4 + x^2y^2 + \frac{1}{2}y^4 + x^4y + \frac{2}{3}x^2y^3 - \frac{1}{5}y^5 + \frac{1}{5}x^6 + x^4y^2 + \frac{1}{3}x^2y^4 + \frac{11}{45}y^6. \quad (6.7)$$

The second-order truncation of this potential is integrable, consisting simply of two linear harmonic oscillators. The third-order truncation is seen to be the Hénon-Heiles potential, which is nonintegrable. In fact, it has been proven [47] that all truncations of the periodic three-particle Toda lattice above second-order are nonintegrable and include both regular and stochastic orbits. The specific case of the sixth-order truncation of this Toda lattice will also be studied.

The truncations of the Toda lattice present relatively similar shadowing problems in that the potentials only involve polynomials. In contrast, shadowing of the anisotropic Kepler potential will also be attempted. In normalized cartesian coordinates, the Hamiltonian for the anisotropic Kepler potential in two dimensions is given by

$$H = \frac{u^2}{2\mu} + \frac{v^2}{2\nu} - \frac{1}{\sqrt{x^2 + y^2}}, \quad (6.8)$$

which corresponds to a system with no angular momentum, and where u and v are the momenta. Besides the energy, this system is also characterized by another free parameter, the mass ratio μ/ν .

Shadowing of the anisotropic Kepler problem is more difficult than the previous problems studied primarily due to the singularity at the origin. Because of this, the bounds required to calculate the quantities in the nonhyperbolic shadowing theorem are no longer defined; hence, a new modification of the theorem is required.

6.2 Shadowing Time Results for Maps

For all of the systems discussed in section 6.1, the shadowing time was calculated given the noise amplitude for a large number of initial conditions. These numerical calculations involved computing the quantities associated with the nonhyperbolic shadowing theorem discussed in chapter 5 until the values for C_n and D_n exceeded their bounds, at which point the shadowing theorem breaks down and the trajectory is no longer shadowable. The locations in phase space where this happens are the homoclinic near-tangencies of the system, hereafter called ‘glitches.’ Recall from chapter 5 that the shadowing time is the length of time that the trajectory is computed before a glitch is encountered. For each noise amplitude δ , the location of the glitch and the shadowing time were determined for an ensemble of 1000 randomly chosen initial conditions. For all systems, noise amplitudes were chosen to be from $\delta = 10^{-14}$ to $\delta = 10^{-7}$. Since all important quantities were calculated in double precision, $\delta = 10^{-14}$, which roughly corresponds to double precision error, is the lower limit on the noise amplitude computationally.

The first system studied was the Hénon map. The shadowing time was computed for each of the 1000 randomly chosen initial conditions for each noise level. Figure 6.3 shows the distribution of shadowing times for representative

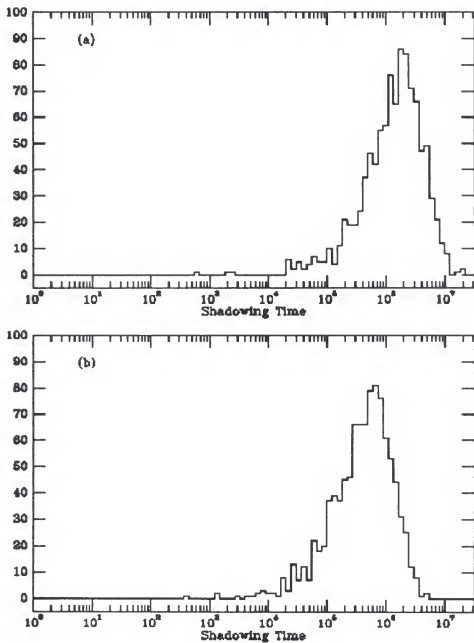


Figure 6.3 Histograms showing the distribution of shadowing times for randomly chosen ensembles of 1000 initial conditions, chosen to lie within 10^{-5} of the origin, for the Hénon map. (a) $\delta = 10^{-14}$; (b) $\delta = 10^{-13}$;

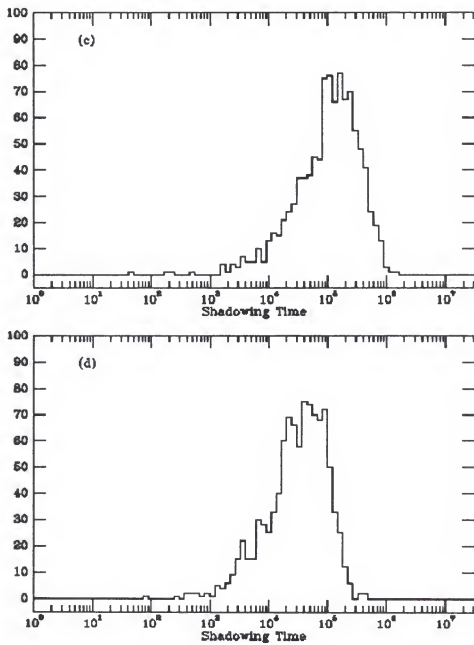


Figure 6.3. (continued)

(c) $\delta = 10^{-12}$; (d) $\delta = 10^{-11}$;

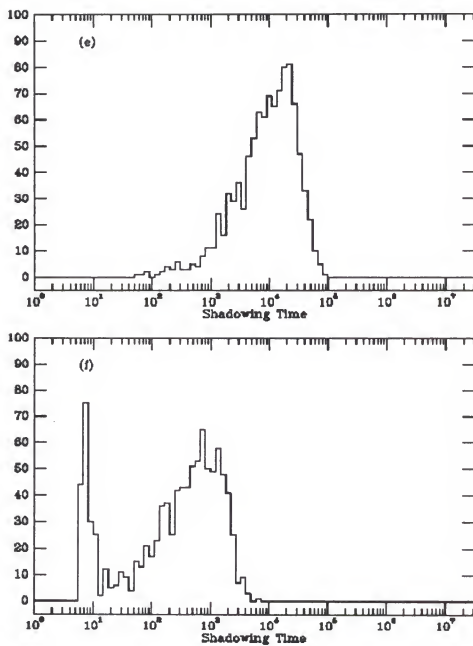


Figure 6.3. (continued)

(e) $\delta = 10^{-10}$; (f) $\delta = 10^{-8}$.

ensembles of 1000 initial conditions, chosen to lie within a distance of 10^{-5} to the origin.

The first conclusion that can be inferred from this data is that the shadowing time varies significantly for this system, spanning several orders of magnitude. The conventional wisdom is that the shadowing time for typical nonhyperbolic systems is long enough to consider any deviation from hyperbolicity to be insignificant. However, we can see that even though the mean shadowing time can be long, there is an appreciable number of orbits with relatively short shadowing times, extending down to about two orders of magnitude below the mean.

Second, these distributions remain essentially identical if one chooses a different ensemble of initial conditions, for example, by grouping the initial conditions around a randomly chosen location instead of the origin, or changing the range around the location about which the initial conditions are chosen to lie.

This last observation is of extreme importance; it seems to imply that the dependence of shadowing time on initial conditions is fractal. In other words, the distributions have similar structure on all scales. Choosing the initial conditions randomly within a region 10^{-10} from the origin yields statistically near-identical distributions as choosing the initial conditions within a region 10^{-5} from the origin. This conclusion is supported by results of the Kolmogorov-Smirnoff test, which shows that these sample distributions have about a 95 percent chance of being drawn from the same distribution.

By comparing the sample distributions in figure 6.3, it is apparent that the mean shadowing time decreases as the noise level is increased. This can easily

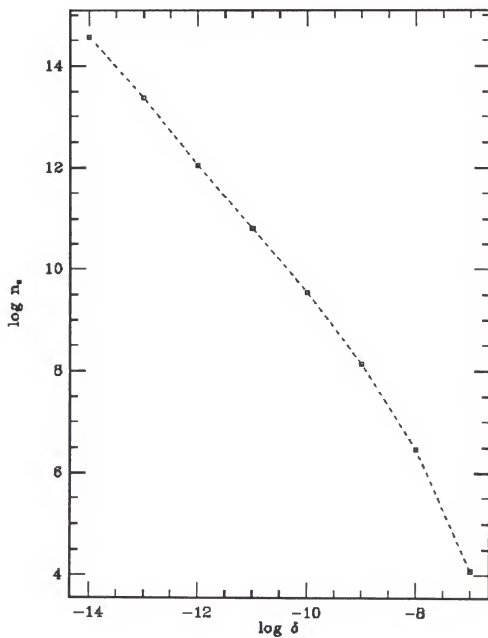


Figure 6.4 $\log \bar{n}_s$ versus $\log \delta$ for the Hénon map, where $\bar{n}_s$ is the mean shadowing time and δ is the noise level. The error bars, obtained by averaging over five distributions, are smaller than the squares.

be understood – increasing the noise level increases the range of angles which are close enough to zero to create a homoclinic near-tangency, making the orbit experience these glitches more frequently. The relationship between the mean shadowing time, $\bar{n}_s$, and the noise level δ is shown in figure 6.4. Each point was determined by averaging over five distributions. Since the distributions are so close to being identical statistically, the standard deviation of each point is smaller than the squares used for plotting.

Hammel, Yorke and Grebogi predict a power law relationship between the shadowing time and the noise amplitude, as provided in the following conjecture [38]:

Conjecture. *For a typical dissipative map $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ with a positive Lyapunov exponent and a small noise amplitude $\delta > 0$, we expect to find $\varepsilon \leq \sqrt{\delta}$ for an orbit length $N \simeq 1/\sqrt{\delta}$.*

This conjecture is not very meaningful when applied to individual orbits, due to the fractal distributions of shadowing times. However, if one interprets the conjecture instead as applying to the mean shadowing time, then it implies effectively that $\bar{n}_s \sim 1/\sqrt{\delta}$ for dissipative maps.

As seen in figure 6.4, The six data points for $\delta = 10^{-14}$ through $\delta = 10^{-9}$ are well fit by a power law $\bar{n}_s \sim \delta^p$, where $p = -0.556 \pm 0.006$. Since the standard deviation for each point is insignificant, the error estimate for the exponent p is simply the formal error associated with the least squares fit of the average values plotted in figure 6.4.

These results roughly support the scaling conjectured by Hammel, Yorke and Grebogi, and a very precise scaling law is evident, though it is not exactly equal to the law $\bar{n}_s \sim 1/\sqrt{\delta}$ predicted, at least within the range of noise

amplitudes $\delta = 10^{-14}$ through $\delta = 10^{-9}$. If one were to include the data points for $\delta = 10^{-8}$ and $\delta = 10^{-7}$, the exponent would be $p = -0.624 \pm 0.030$. This trend towards slightly shorter shadowing times than expected for high noise amplitude appears to be due to the 'spikes' in the histograms for these noise amplitudes.

The fractal nature of the shadowing time distributions appear to be a universal feature, which is not surprising. Similar distributions are observed in the standard map, and are shown in figure 6.5 for the case that $K = 7$. At large values of K , the measure of the regular region goes to zero. Thus, this value of K was chosen to ensure that all orbits would be within the stochastic sea, and that there would be no effect due to choosing initial conditions close to regular regions of phase space. Again, these distributions are independent of the range and location of the chosen initial conditions. These results are gain supported by the Kolmogorov-Smirnoff test, which verifies that these sample distributions have a probability of within 90% of being drawn from the same distribution. Similar results are obtained for the value $K = 3$.

A conjecture similar to that for dissipative maps has been reported by Grebogi et al [39] for Hamiltonian maps:

Conjecture. *For a typical two-dimensional Hamiltonian map yielding chaotic trajectories with a small noise amplitude $\delta > 0$, we expect to find $\varepsilon \leq \sqrt{\delta}$ for a trajectory of length $N \simeq 1/\sqrt{\delta}$.*

The relationship between the mean shadowing time, $\bar{n}_s$, and the noise level δ is shown in figure 6.6. Again, the data points were obtained by averaging $\bar{n}_s$ over five different ensembles of 1000 initial conditions each for each noise level. Fitting the data points to a straight line yields $\bar{n}_s \sim \delta^p$, where $p = -0.498 \pm$

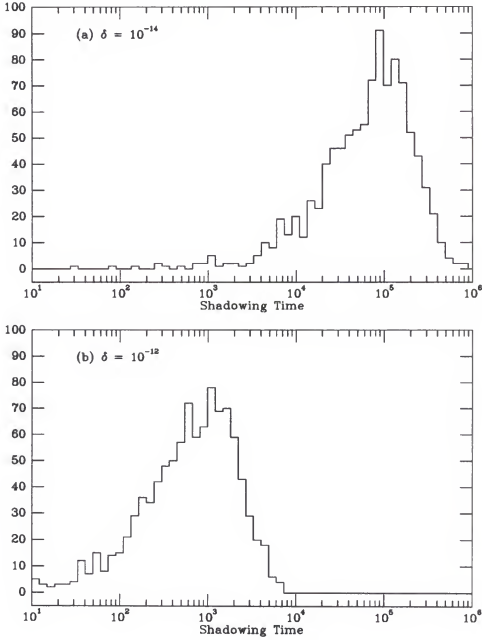


Figure 6.5 Histograms showing the distribution of shadowing times for randomly chosen ensembles of 1000 initial conditions, chosen to lie within 10^{-5} of the origin, for the standard map with $K = 7$. (a) $\delta = 10^{-14}$; (b) $\delta = 10^{-12}$;

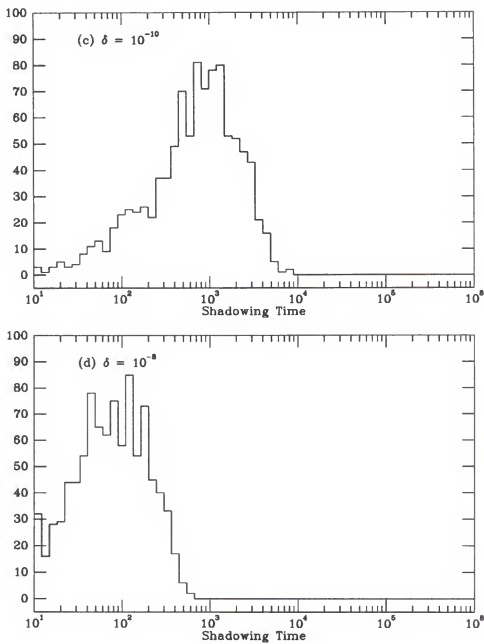


Figure 6.5. (continued)

(c) $\delta = 10^{-10}$; (d) $\delta = 10^{-8}$.

0.002, in complete agreement with the conjecture given the interpretation that the trajectory length should be the mean shadowing time.

6.3 Shadowing Time Results for Hamiltonian Differential Equations

Similar shadowing time distributions are observed for the case of chaotic Hamiltonian differential equations, as observed for the Hénon-Heiles potential (figure 6.7) given an energy $E = 0.16$ and for the sixth-order truncation of the Toda Lattice (figure 6.8) for an energy of $E = 30$. In each case, the energy was chosen to yield a significant measure of stochastic orbits. For the Hénon-Heiles potential, results were obtained for an energy of $E = 0.14$ as well, with statistically similar results.

The shadowing method for two degree-of-freedom Hamiltonian systems is similar to the method used for two-dimensional maps. The primary difficulty is due to the calculation of the stable and unstable manifolds. A significantly larger number of time steps is required for the manifolds to converge for the case of differential equations and, in fact, they are often observed not to converge at all, in a reasonable amount of time. This is indeed the case for the sixth-order truncation of the Toda lattice; if the energy is increased much beyond $E = 30$, the phase space becomes more and more stochastic, and the stable manifold is not easily computed. Even for $E = 30$, many choices of initial conditions did not yield reliable results due to the inability to calculate the stable manifold, and so the distributions shown in figure 6.8 are for a smaller number of initial conditions.

While generic two degree-of-freedom systems have a four-dimensional phase space, two degree-of-freedom Hamiltonian systems have only one-dimensional stable and unstable manifolds. Due to Liouville's theorem, the number of stable

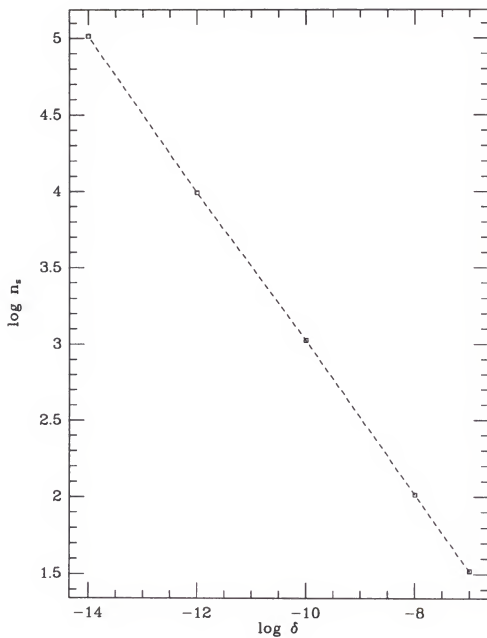


Figure 6.6 $\log \bar{n}_s$ versus $\log \delta$ for the standard map with $K = 7$. The error bars, obtained by averaging over five distributions, are smaller than the squares.

(contracting) directions must equal the number of unstable (expanding) directions, to preserve phase space volume. Since $E = \text{const.}$ can be used to reduce the phase space dimension by one, and the direction along the trajectory can be neither expanding nor contracting, the stable and unstable manifolds can each have only one dimension. The numerical method of calculating shadowing times in this case is shown in appendix A. If the system is not Hamiltonian, these manifolds are not necessarily one-dimensional and, in fact, the dimension of the manifolds may actually change with time. This case is considered by Dawson et al [48], who determine that the shadowing time can actually be much lower in this case, due to the fluctuating number of dimensions of the invariant manifolds. The calculation of shadowing times for two-dimensional stable and unstable manifolds can be done using the Fortran code shown in appendix B.

Unlike the case for maps, the relation between the mean shadowing time and the noise amplitude is not a simple power law for these potentials. This is shown in figure 6.9 and figure 6.10. This should not be too surprising, since there is no a priori reason why this relation should be a power law. Recall that a ‘glitch’ occurs when the angle between the stable and unstable manifolds is small enough that the noise amplitude is able to cause a trajectory to ‘jump over’ a stable manifold. It is therefore apparent that the mean shadowing time should be a monotonically decreasing function of the noise amplitude. There is no guarantee, however, that this relationship be a simple function, e.g. a power law. Indeed, as a particle moves along a trajectory, the angle between the stable and unstable manifolds fluctuates smoothly in the case of continuous systems, but the angle is not necessarily a monotonic function of time. As this

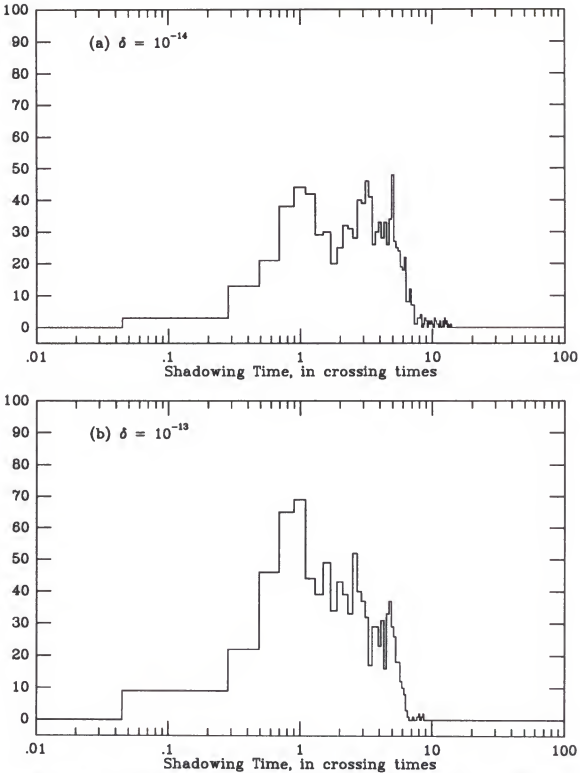


Figure 6.7 Histograms showing the distribution of shadowing times for randomly chosen ensembles of 1000 initial conditions, chosen to lie within 10^{-5} of the origin, for the Hénon-Heiles potential with $E = 0.16$. (a) $\delta = 10^{-14}$; (b) $\delta = 10^{-13}$;

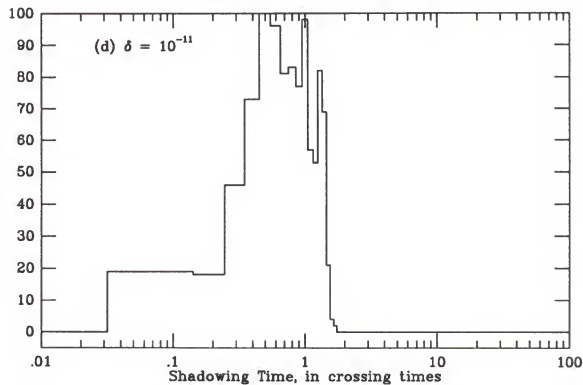
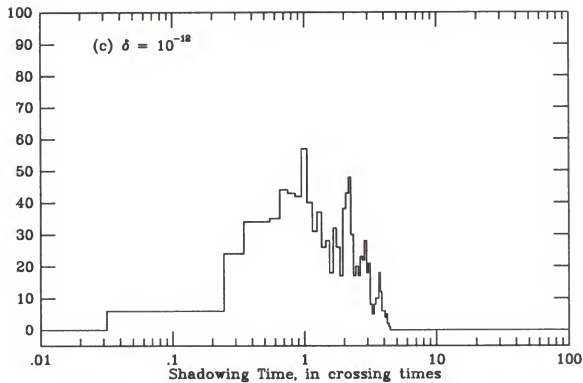


Figure 6.7. (continued)

(c) $\delta = 10^{-11}$; (d) $\delta = 10^{-10}$.

angle fluctuates, it then has several maxima and minima; this causes certain noise amplitudes to intersect the angle versus time curve more frequently than others.

The power law conjectures of Grebogi, Hammel, Sauer and Yorke actually hold up quite well in the case of these continuous systems if only a smaller range of noise amplitudes is considered, for example, from $\delta = 10^{-12}$ to $\delta = 10^{-10}$ in these cases. For very low noise amplitudes, however, the shadowing times are considerably shorter than one would predict from the scaling law $\bar{n}_s \sim \delta^p$.

6.4 Results Concerning Glitch Locations

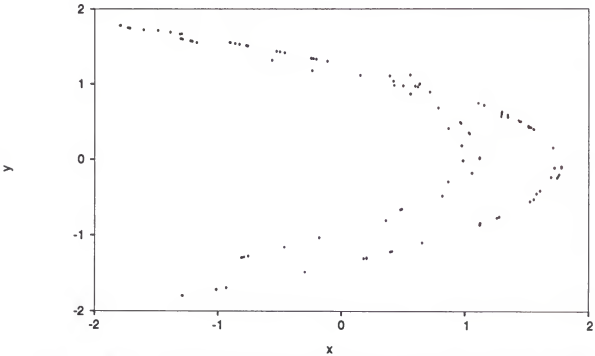


Figure 6.11 The glitch locations for the Hénon map for an ensemble of 1000 initial conditions and $\delta = 10^{-14}$.

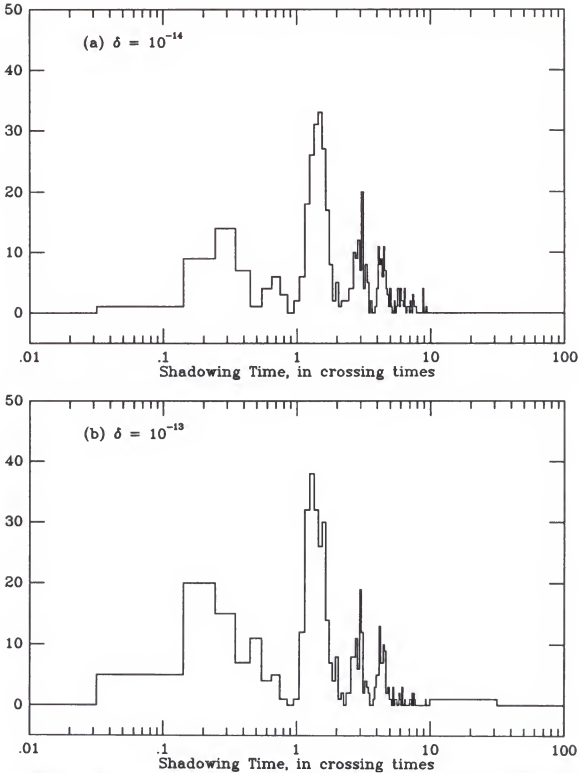


Figure 6.8 Histograms showing the distribution of shadowing times for randomly chosen ensembles of 1000 initial conditions, chosen to lie within 10^{-5} of the origin, for the sixth-order Toda potential with $E = 30$. (a) $\delta = 10^{-14}$; (b) $\delta = 10^{-13}$;

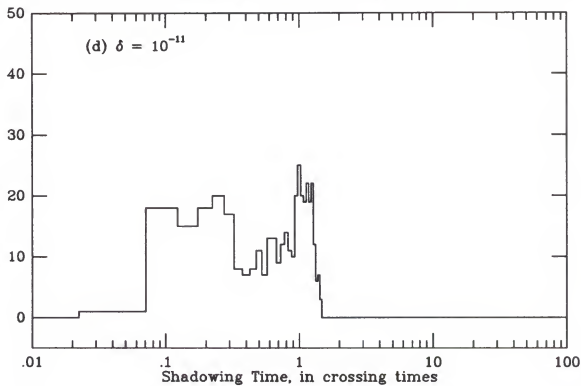
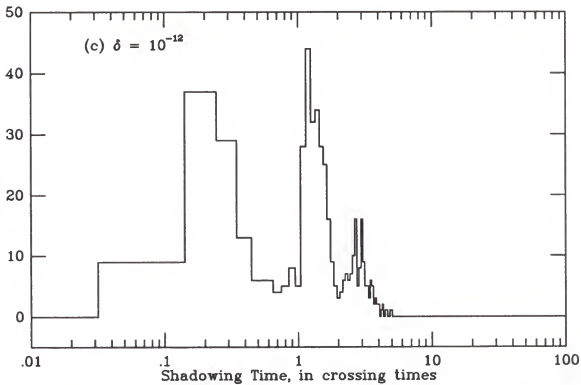


Figure 6.8. (continued)

(c) $\delta = 10^{-12}$; (d) $\delta = 10^{-11}$.

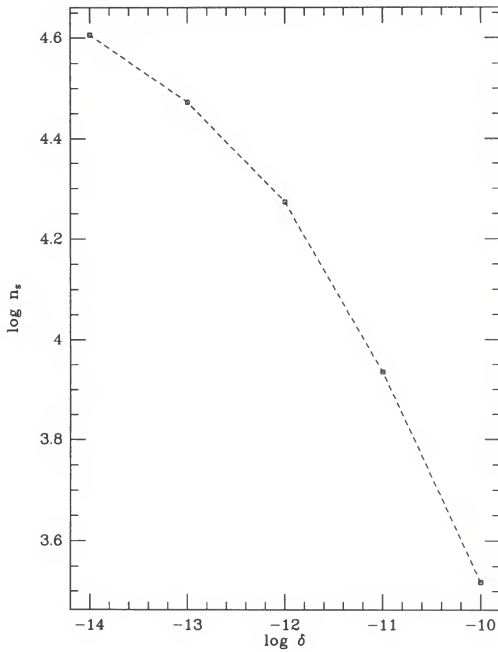


Figure 6.9 $\log \bar{n}_s$ versus $\log \delta$ for the Hénon-Heiles potential with $E = 0.16$.

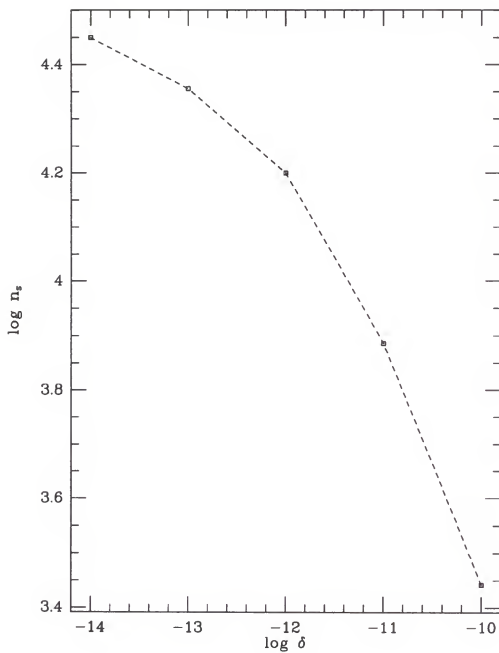


Figure 6.10 $\log \bar{n}_s$ versus $\log \delta$ for the sixth-order truncation of the Toda lattice with $E = 30$.

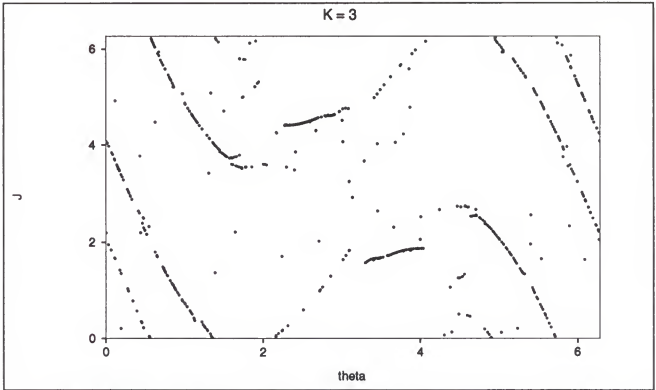


Figure 6.12 The glitch locations for the standard map with $K = 3$ for an ensemble of 1000 initial conditions and $\delta = 10^{-14}$.

Recall that a glitch is defined to be the location at which the stable and unstable manifolds are close enough to being tangent that the trajectory is not shadowable at that point for the given noise amplitude. For each initial condition, the glitch locations were determined as well as the shadowing time. For the Hénon map, the locations of the glitches for one representative ensemble of initial conditions with $\delta = 10^{-14}$ are shown in figure 6.11. It should be apparent that, if one were to continue computing the locations of these near-tangencies, the resulting figure would not be distinguishable from the Hénon attractor itself, figure 6.1. Thus, it appears that the homoclinic near-tangencies are dense within the attractor. Similar results are found for other choices of the noise level.

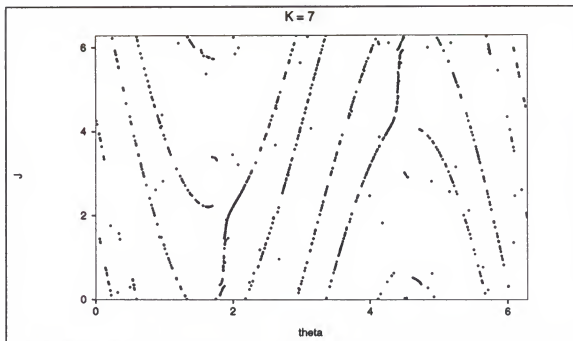


Figure 6.13 The glitch locations for the standard map with $K = 7$ for an ensemble of 1000 initial conditions and $\delta = 10^{-14}$.

The situation is somewhat different for area-preserving maps, which do not have attractors. In figure 6.12, the glitch locations are shown for the standard map with $K = 3$. As can be seen, these glitches are not randomly distributed throughout the stochastic region of phase space, but appear to lie along distinct curves within the stochastic region. Comparing the glitch locations to the stochastic region of phase space observed in figure 6.2, it is observed that a number of these glitches occur at the boundary of the stochastic and regular regions.

A similar distribution of glitches occurs for $K = 7$ as well, as shown in figure 6.13, even though, as shown in figure 6.14, practically the entire phase space appears stochastic.

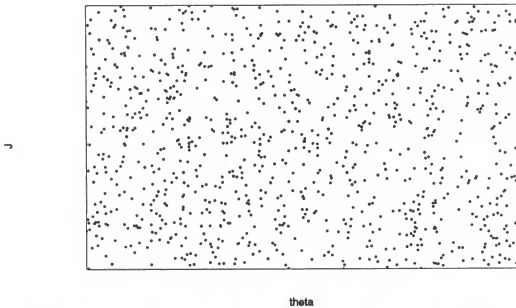


Figure 6.14 A single stochastic orbit for the standard map with $K = 7$.

CHAPTER 7

DISCUSSION AND CONCLUSIONS

This chapter is devoted to a recapitulation of the major results of this study, as well as some general remarks concerning the interpretation of these results.

The results of this study seem to indicate that, even for gravitational systems with a large number of particles, discrete effects due to close encounters between particles have an important effect on the instability of trajectories in the system. These results hold provided that the system of interest is nonintegrable and allows for a finite fraction of stochastic orbits. Numerical studies seem to indicate that this is in fact the case for elliptical galaxies.

These discreteness effects are seen to increase the rate of instability in gravitational N -body systems, as discussed in chapter 3. Results of these numerical studies indicate that changes in the softening parameter, which was used to alter the effect of close encounters, certainly changed the rate of instability. There is no sense in which the instability "turns off" when N is large, and close encounters remain important. Our numerical studies hence indicate that, even for a large number of stars, both the effects of the smooth mean field and the close encounter noise are important when determining the rate of instability in gravitational N -body systems.

Analytical studies also indicate how a nonintegrable mean field and noise due to discreteness effects are both important when determining the rate of

instability. The primary conclusion of chapter 4 is that the presence of a nonintegrable mean field drastically alters the relaxation time, which was calculated by Chandrasekhar in the idealised case that, without collisions, the stars follow a smooth, rectilinear trajectory. These new calculations indicate that the exponential instability of stellar trajectories in the nonintegrable mean field reduces the relaxation time significantly.

The nonhyperbolic shadowing theorem was used to determine how large an effect noise of a given amplitude may have, given that, in most physically relevant situations, the nonintegrable mean field is nonhyperbolic. Results from both maps and differential equations support the claim that systems which include dynamical noise can be shadowed for long times for *most* orbits, provided that the noise amplitude δ is small enough. However, the existence of orbits with appreciably *shorter* shadowing times should cause concern when integrating chaotic dynamical systems for long times. The existence of these 'unshadowable' orbits makes conclusions about the 'shadowability' of orbits in a nonhyperbolic system unclear, even though the mean shadowing time may be quite large. This is especially true considering the fractal nature of the sample distributions of shadowing times. If one is interested in the shadowing time for an orbit with a certain initial condition, one has to realize that there are orbits with initial conditions *arbitrarily close* to the desired initial condition with drastically different shadowing times.

However, nonhyperbolic shadowing may be used to determine the fraction of orbits that may be qualitatively changed by the presence of noise. Even given the fractal nature of the shadowing times, the sample distributions yield

information as to how many orbits are ‘unshadowable’ given the number of iterations of the map.

As an example, consider the situation in which one is concerned with how computer errors are going to affect orbits in a deterministic system. We can see from the histograms in chapter 6 that, even for double precision calculations, the majority of orbits in the Hénon-Heiles potential and sixth-order Toda lattice truncation are unshadowable after only ten crossing times, and several are unshadowable after less than one crossing time. In the case of the Hénon-Heiles potential, the fraction of orbits that become unshadowable in less than a crossing time is about ten percent. Thus, any orbit that behaves peculiarly, even within the first crossing time, should be considered suspect. Since these distributions appear fractal, it is impossible to determine if such unusual orbits are actual orbits of the deterministic system, or are associated with a breakdown in shadowing, even in principle.

It remains to be seen whether ‘unshadowability’ of orbits actually affects the statistical properties of the orbits involved, however. It is certainly true that if a system is *shadowable*, then it is structurally stable with respect to noise. In fact, hyperbolic systems have been proven to be structurally stable by Anosov [49]. The converse of this statement, however, is not necessarily true. In fact, for the standard map with large stochasticity ($K = 7$), the existence of homoclinic near-tangencies does not appear to alter the statistical properties of the orbits. This would seem to imply that the standard map for $K = 7$ is *structurally stable* in the sense that the statistical properties of the map is unaffected by the presence of noise. It remains to be seen whether this behavior is universal. For example, it has been observed that even weak

noise can affect the near-invariant measure of Hamiltonian potentials [50]. One would expect that this behavior is due to the existence of homoclinic near-tangencies within the system.

The fact that the glitches in the standard map are not randomly distributed would seem to imply that the location of the homoclinic near-tangencies may be related to some other property of the dynamical system, for example, the Lyapunov exponents. The existence of glitches adjacent to KAM tori may be associated with 'sticky' orbits with smaller local Lyapunov exponents than are found in the stochastic sea. These sticky orbits have been observed for the standard map [51] and for several two-dimensional potentials as well [52]. This may have important consequences; orbits which experience a glitch adjacent to invariant tori or cantori may perhaps 'jump' across the barrier, even if a purely deterministic orbit would not. These glitches would then serve as the mechanism for diffusion across these barriers in noisy dynamical systems.

It is important to note that the results concerning the shadowing times do not seem to depend on the form of noise present. It is certainly true that for hyperbolic systems the form of noise does not really matter when determining the shadowability of the system; the form of noise will alter the details of individual orbits, but the statistical properties of a hyperbolic system will remain unaltered, since they are structurally stable in the presence of noise. For nonhyperbolic systems, however, the situation is more complicated.

The shadowing results mentioned in the previous chapter appear to be independent of the form of noise; only the amplitude of the noise was considered to be a factor. The important thing to realize is that the shadowing time must be interpreted as an upper bound on the effect the noise will have; A noisy orbit

is not guaranteed to be affected at the homoclinic near-tangency it encounters; it is only guaranteed not to be affected if it does not approach a near-tangency. A certain form of noise may have less of an effect than this bound indicates; however, as long as the amplitude of the noise is given, the shadowing theorem guarantees that the choice of a certain form of noise will not have more of an effect than indicated.

It could also be argued that, even if an orbit is unshadowable, the statistical properties of the system may remain unaffected, as it appears for the standard map. Assume a noisy orbit encounters a homoclinic near-tangency at time step N , after which the orbit no longer is shadowed by the original deterministic orbit. This orbit will then, in all probability, begin to be shadowed by a new deterministic orbit beginning at time step $N + 1$. Despite the fact that this noisy orbit will not be similar to any purely deterministic orbit of the system, statistical tests will most likely indicate no difference between the noisy and deterministic orbits.

The only situation for which a noisy orbit will not yield qualitatively similar results to what is expected is when the orbit passes between a stochastic region of phase space and a regular region, i.e. that it crosses KAM tori in the process. In this case, the qualitative behavior of the orbit is affected. It may be expected that this should not happen very often for systems with global stochasticity, in which most orbits are well within the stochastic sea, far away from any regular regions remaining. However, as observed in the case of the standard map, the location of homoclinic near-tangencies seem to be correlated with the presence of KAM tori, or, at least, cantori. Thus, the effect of these near-tangencies may be more pronounced than how it originally appears.

APPENDIX A

FORTRAN CODES - TWO-DIMENSIONAL SYSTEMS

```

Program Shadow
*
* This program calculates the shadowing time and location of glitch
* for a two degree-of-freedom Hamiltonian differential equation,
* specifically, the sixth-order truncation of the Toda lattice.
* A list of 1000 initial conditions is randomly determined and
* stored in a file. The shadowing times for each initial condition
* is then stored in a file, and the location of the glitch for each
* initial condition is stored in a third file. The map, its
* Jacobian, and the inverse of the Jacobian must be input as
* separate subroutines.
*
* Variables --
*   x1,y1,xd1,yd1 -- the initial conditions
*   x2,y2,xd2,yd2 -- the glitch location
*   ran2 -- random number function, from Numerical Recipes
*   idum -- random number seed
*   i -- index to calculate ith initial condition
*   ltime -- the shadowing time
*   lreg -- Warning flag for regular or escaping point
*
* Files --
*   file1 -- the initial conditions
*   file2 -- the initial conditions and the shadowing time, or
*           a warning statement
*   file3 -- the glitch location
*   todadat -- the three new files, followed by the model
*             parameters
*
* Double precision x1,y1,xd1,yd1,x2,y2,xd2,yd2
* double precision sigma
* double precision delta,E,dimen
* Real ran2
* Integer i,jran,lreg,ltime
* data idum/-3303039/
* character*12 file1,file2,file3
* common // delta,E,dimen
* open(unit=25,file='todadat',status='old')
* read(25,*)file1,file2,file3
* open (unit=18,file=file1,status='new')
* open (unit=19,file=file2,status='new')
* open (unit=20,file=file3,status='new')
* open (unit=21,file='smfld.dat',status='old')
* open (unit=22,file='umfld.dat',status='old')
* open (unit=23,file='coords.dat',status='old')
*
* Initialize and write initial conditions.
*
*   read(25,*)delta,E,dimen
*   do i=0,999
*     x1=0.d0

```

```

      y1=ran2(idum)
      yd1=ran2(idum)
      xd1=dsqrt(2.d0+E-yd1*yd1-y1*y1+2.d0*y1*y1+1/3.d0
!         -y1**4+2.d0*y1**5/3.d0-22.d0*y1**6/45.d0)
      write(18,*)x1,y1,xd1,yd1
    end do
    rewind (unit=18)

*
* For each initial condition, call the main subroutine, and store
* results.
*
    do ii=1,1000
      ltime=0
      lreg=0
      read(18,*)x1,y1,xd1,yd1
      call mflld(x1,y1,xd1,yd1,x2,y2,xd2,yd2,ltime,lreg,sigma)
      if (lreg .eq. 1)then
        ltime=0
      endif
      write(19,*)ii,x1,yd1,ltime,lreg
      write(20,*)x2,y2,xd2,yd2
    end do
    close(unit=18,status='delete')
  end

  Subroutine mflld(x1,y1,xd1,yd1,x2,y2,xd2,yd2,ltime,lreg,sigma)

*
* Given the initial conditions x1,y1,xd1,yd1, compute the shadowing
* time and the location of the glitch for the sixth-order trunc-
* cation of the Toda lattice. Return these values or an error flag
* to the driver program. The noise level delta and the energy must
* be specified.
*
* Variables --
*   x(n),y(n),xd1(n),yd1(n) -- The phase space variables at time
*   step n
*   z -- swap storage used when computing x(n+1),y(n+1)
*   ajac(i,j) -- the jacobian elements of the standard map
*   ajacinv(i,j) -- the inverse jacobian elements
*   anorm(i) -- the vector used to normalize
*   u(i,n),s(i,n) -- the stable and unstable manifolds, ith
*   component, at time step n
*   csc(n) -- the cosecant of the angle at time step n
*   cscc -- the cosecant for the current time step
*   u1,u2,s1,s2 -- the two elements of the stable and unstable
*   manifolds at the current time step
*   rn(n),tn(n) -- the bounds on the expansion of the map at
*   time step n
*   cn(n),dn(n) -- the recursive variables, at time step n
*   rai,tni,cni,dni -- the above variables for a single time
* step
*   xin, yin -- the action and angle variables at the current
*   time step, passed to subroutine map
*   xout, yout -- the variables at the next time step, output
*   from subroutine map
*   abnd -- The bound on cn and dn, from the theorem
*   en(n) -- the same as dn(n), but backwards
*   delta -- the noise level
*   E -- the energy
*   x1,y1,xd1,yd1 -- the phase space variable initial
* conditions
*   dimen -- the dimension of the system
*   uone(n),utwo(n) -- two choices for the unstable manifold,
*   which must converge
*   sone(n),stwo(n) -- two choices for the stable manifold
*   udiff1,udiff2 -- the convergence criterion for the manifolds
*   ee -- used in determining of e(n) approaches bound of theorem

```

```

*      1 -- flag used to determine end of shadowability
*
Double Precision x(500001),y(500001),z,ajac(4,4),ajacinv(4,4)
Double Precision sone(4,500001),csc(10000),csec
Double Precision u1(4),u2(4),s1(4),s2(4),rn(10000),tn(10000)
Double Precision cn(10000),dn(10000),bnd,xin,yin,xout,yout
Double Precision abnd,en(10000),csc1,csc2,rni,tni,ci,di
Double Precision bbnd,a,b,x1,y1
Double Precision uone(4,500000)
Double Precision xdin,xdot,ydin,ydot,xdot(500000),ydot(500000)
Double Precision xd1,yd1,x2,y2,xd2,yd2
Double Precision t,x1,y1,xd1,yd1,h
Double Precision x1a(4),dxdt(4),dmdt(4)
Double Precision u1norm,u2norm,s1norm,s2norm
Double Precision anorm(4),aident(4,4)
Double Precision theta,sigma
double precision delta,E,dimen
Real ran2
Integer jran
Integer N,i,k,l,kk,ltime,lreg
data idum/-3303051/
common // delta,E,dimen
N=5000
lreg=0
ltime=0
bnd=2.d0
cn(1)=0.d0
abnd=dimen**(-5.d0/2.d0)*bnd**(-2.d0)/dSqrt(delta)
dn(N)=abnd
en(1)=dn(N)
l=0
ii=0
h=1.d-4
do i=1,4
  do j=1,4
    aident(i,j)=0.d0
  end do
end do
do i=1,4
  aident(i,i)=1.d0
end do
read(21,*)i,s1(1),s1(2),s1(3),s1(4),lreg
read(22,*)i,u1(1),u1(2),u1(3),u1(4),lreg
read(23,*)x1,y1,xd1,yd1
if ((lreg .eq. 1) .or. (lreg .eq.2)) then
  return
endif
write(6,*)'stable mfld'
write(6,*)x1,y1,xd1,yd1
write(6,*)s1(1),s1(2),s1(3),s1(4)
do j=1,4
  uone(j,1)=u1(j)
  sone(j,5001)=s1(j)
end do
x(1)=x1
y(1)=y1
xdot(1)=xd1
ydot(1)=yd1

*      Begin main loop to calculate shadowing times, locations.
*
*      Do kk=1,20000
*
*      Calculate the unstable manifold for the next N time steps.
*
  Do i=1,N
    xi=x(i)

```

```

        yi=y(i)
        rdi=xdot(i)
        ydi=ydot(i)
        anorm(1)=uone(1,i)
        anorm(2)=uone(2,i)
        anorm(3)=uone(3,i)
        anorm(4)=uone(4,i)
    Call Jacob(ajac,xi,yi)
    do k=1,4
        do j=1,4
            ajac(k,j)=ajac(k,j)*h+aident(k,j)
        end do
        anorm(k)=ajac(k,1)*u1(1)+ajac(k,2)*u1(2)+
!           ajac(k,3)*u1(3)+ajac(k,4)*u1(4)
    end do
    Call Normal(anorm)
    uone(1,i+1)=anorm(1)
    uone(2,i+1)=anorm(2)
    uone(3,i+1)=anorm(3)
    uone(4,i+1)=anorm(4)
    anorm(1)=x(i)
    anorm(2)=y(i)
    anorm(3)=xdot(i)
    anorm(4)=ydot(i)
    call Taylor(anorm,h)
    x(i+1)=anorm(1)
    y(i+1)=anorm(2)
    xdot(i+1)=anorm(3)
    ydot(i+1)=anorm(4)
    if ((dabs(x(i+1)) .gt. 1.d03) .or. (dabs(y(i+1)) .gt. 1.d03))
then
        lreg = 2
        return
    endif
End do
*
* Calculate the stable manifold for the next N time steps.
*
    Do i=N,1,-1
        xi=x(i)
        yi=y(i)
        rdi=xdot(i)
        ydi=ydot(i)
        anorm(1)=sone(1,i+1)
        anorm(2)=sone(2,i+1)
        anorm(3)=sone(3,i+1)
        anorm(4)=sone(4,i+1)
        Call Jacinv(ajacinv,xi,yi)
        do k=1,4
            do j=1,4
                ajacinv(k,j)=ajacinv(k,j)*h+aident(k,j)
            end do
            anorm(k)=ajacinv(k,1)*s1(1)+ajacinv(k,2)*s1(2)+
!           ajacinv(k,3)*s1(3)+ajacinv(k,4)*s1(4)
        end do
        Call Normal(anorm)
        sone(1,i)=anorm(1)
        sone(2,i)=anorm(2)
        sone(3,i)=anorm(3)
        sone(4,i)=anorm(4)
    End do
*
* Given the stable and unstable manifolds, determine the important
* shadowing quantities.
*
    Do i=1,N
        do j=1,4

```

```

    u1(j)=uone(j,i)
    s1(j)=sone(j,i)
  end do
  Call angle(u1,s1,csec,theta)
  csc(i)=csec
  do j=1,4
    u1(j)=uone(j,i+1)
    s1(j)=sone(j,i+1)
  end do
  Call angle (u1,s1,csec,theta)
  csc(i+1)=csec
  xin=x(i)
  yin=y(i)
  do j=1,4
    u1(j)=uone(j,i+1)
    s1(j)=sone(j,i)
  end do
  xi=x(i-1)
  yi=y(i-1)
  xdi=xdot(i-1)
  ydi=ydot(i-1)
  Call Jacob(ajac,xi,yi)
  Call Jacinv(ajacin,xi,yi)
  Call bound(ajac,ajacin,u1,s1,rni,tni,h,aident)
  rn(i)=rni
  tn(i)=tni
  ci=cn(i)
  di=en(i)
  csc1=csc(i)
  csc2=csc(i+1)
  Call recur(csc1,csc2,rni,tni,ci,di,abnd,l,ltime)
  if (l .eq. 1) then
    x2=x(i-1)
    y2=y(i-1)
    xd2=xdot(i-1)
    yd2=ydot(i-1)
    do j=1,4
      anorm(j)=uone(j,i-1)
    end do
    return
  end if
  cn(i+1)=ci
  en(i+1)=di
End do

*
* If glitch has not yet been encountered, re-do process for the next
* N time steps.
*
  x(1)=x(N+1)
  y(1)=y(N+1)
  xdot(1)=xdot(N+1)
  ydot(1)=ydot(N+1)
  en(1)=en(N+1)
  cn(1)=cn(N+1)
  anorm(1)=uone(1,N+1)
  anorm(2)=uone(2,N+1)
  anorm(3)=uone(3,N+1)
  anorm(4)=uone(4,N+1)
  Call normal(anorm)
  uone(1,1)=anorm(1)
  uone(2,1)=anorm(2)
  uone(3,1)=anorm(3)
  uone(4,1)=anorm(4)
  x1=x(1)
  y1=y(1)
  xd1=xdot(1)
  yd1=ydot(1)

```

```

ii=20000
call smfld(x1,y1,xdi,ydi,s1,lreg,ii)
if ((lreg .eq. 1) .or. (lreg .eq.2)) then
  return
endif
sone(1,N+1)=s1(1)
sone(2,N+1)=s1(2)
sone(3,N+1)=s1(3)
sone(4,N+1)=s1(4)
end do
Write(6,*) 'map can be shadowed for over 100 million iterates.'
End

```

Subroutine Normal(anorm)

- * The four-dimensional vector anorm is normalized.

```

Double Precision anorm(4),w
w=DSqrt(anorm(1)**2+anorm(2)**2+anorm(3)**2+anorm(4)**2)
anorm(1)=anorm(1)/w
anorm(2)=anorm(2)/w
anorm(3)=anorm(3)/w
anorm(4)=anorm(4)/w
Return
End

```

Subroutine angle(u,s,csec,theta)

- * The angle between the stable and unstable manifolds at a given time
- * step is calculated for one-dimensional manifolds in a four-dimensional
- * space.

```

Double Precision u(4),s(4),csec
Double Precision theta,dot,normu,norms,alpha
dot=0.d0
normu=0.d0
norms=0.d0
do i=1,4
  dot=dot+u(i)*s(i)
  normu=normu+u(i)**2
  norms=norms+s(i)**2
end do
normu=DSqrt(normu)
norms=DSqrt(norms)
alpha=dot/(normu*norms)
if (alpha .gt. 1.d0) then alpha = 1.d0
theta=dacos(alpha)
if (Dsin(theta) .gt. 0.d0) then
  csec=Dabs(1.d0/Dsin(theta))
endif
Return
End

```

Subroutine bound(ajac,ajacinv,u,s,rn,tn,delt,aident)

- * The bound on the growth of one-dimensional stable and unstable manifolds
- * is calculated.

```

Double precision ajac(4,4),ajacinv(4,4),u(4),s(4)
Double precision sj(4),uj(4),delt,aident(4,4)
Double precision rn,tn,sigma1,sigma2
do i=1,4
  sj(i)=0.d0
  uj(i)=0.d0
end do
do i=1,4
  do j=1,4
    sj(i)=sj(i)+(ajac(i,j)*delt+aident(i,j))*s(j)
    uj(i)=uj(i)+(ajacinv(i,j)*delt+aident(i,j))*u(j)
  end do
end do
rn=DSqrt(sj(1)**2+sj(2)**2+sj(3)**2+sj(4)**2)

```

```

tn=Dsqrt(uj(1)**2+uj(2)**2+uj(3)**2+uj(4)**2)
return
end

```

```

Subroutine recur(csc1,csc2,rni,tni,ci,ei,abnd,l,ltime)
* This subroutine calculates the recursive variable ci and ei for a
* four-dimensional system, given the angle between the stable and unstable
* manifolds, the bounds on the growth of these manifolds, and the
* bound of the shadowing theorem. The time step is increased during this
* subroutine. If the shadowing theorem bound is surpassed, a flag is
* sent to the main program.

```

```

Double precision csc1,csc2,abnd,en
Double precision rni,tni,ci,ei
ltime=ltime+1
ci=csc2+rni*ci
en=(ei-csc1)/tni
if (en .gt. abnd) then
ei=abnd
else
ei=en
endif
if ((ci .gt. abnd) .or. (ei .lt. 0.0)) then
write(6,*) 'unstable point is no longer shadowable'
write(6,*) ltime
write(6,*) 'csc1,csc2',csc1,csc2
write(6,*) 'rn,tn',rni,tni
write(6,*) 'ci,ei',ci,ei
l=1
endif
return
End

```

```

Subroutine Jacob(ajac,xin,yin)
Double Precision ajac(4,4),xin,yin
if (xin .lt. 1.d-20) then
xin=0.d0
endif
if (yin .lt. 1.d-20) then
yin=0.d0
endif
ajac(1,1)=0.d0
ajac(1,2)=0.d0
ajac(1,3)=1.d0
ajac(1,4)=0.d0
ajac(2,1)=0.d0
ajac(2,2)=0.d0
ajac(2,3)=0.d0
ajac(2,4)=1.d0
ajac(3,1)= -1.d0 - 6.d0*xin**2 - 6.d0*xin**4 - 2.d0*yin -
! 12.d0*xin**2*yin - 2.d0*yin**2 -
! 12.d0*xin**2*yin**2 - 4.d0*yin**3/3.d0 -
! 2.d0*yin**4/3.d0
ajac(3,2)=-2.d0*xin - 4.d0*xin**3 - 4.d0*xin*yin -
! 8.d0*xin**3*yin - 4.d0*xin*yin**2 -
! 8.d0*xin*yin**3/3.d0
ajac(3,3)=0.d0
ajac(3,4)=0.d0
ajac(4,1)=-2.d0*xin - 4.d0*xin**3 - 4.d0*xin*yin -
! 8.d0*xin**3*yin - 4.d0*xin*yin**2 -
! 8.d0*xin*yin**3/3.d0
ajac(4,2)=-1.d0 - 2.d0*xin**2 - 2.d0*xin**4 +
! 2.d0*yin - 4.d0*xin**2*yin - 6.d0*yin**2 -
! 4.d0*xin**2*yin**2 + 20.d0*yin**3/3.d0 -
! 22.d0*yin**4/3.d0
ajac(4,3)=0.d0
ajac(4,4)=0.d0
return

```

end

```

Subroutine Jacinv(ajacinv,xin,yin)
Double Precision ajacinv(4,4),xin,yin,denom
if (xin.lt. 1.d-20) then
  xin=0.d0
endif
if (yin.lt. 1.d-20) then
  yin=0.d0
endif
denom=(1.d0 + 4.d0*xin**2 + 4.d0*xin**4 + 8.d0*xin**6 +
! 12.d0*xin**8 - 8.d0*xin**2*yin - 24.d0*xin**4*yin -
! 16.d0*xin**6*yin + 4.d0*yin**2 + 8.d0*xin**2*yin**2 +
! 40.d0*xin**4*yin**2 - 16.d0*xin**6*yin**2 + 8.d0*yin**3/3.d0
! - 16.d0*xin**2*yin**3 - 80.d0*xin**4*yin**3/3.d0 +
! 4.d0*yin**4 + 40.d0*xin**2*yin**4/3.d0 +
! 152.d0*xin**4*yin**4/3.d0 + 8.d0*yin**5 -
! 16.d0*xin**2*yin**5/3.d0 + 88.d0*yin**6/9.d0 +
! 752.d0*xin**2*yin**6/9.d0 + 16.d0*yin**7/3.d0 +
! 44.d0*yin**8/9.d0)
ajacinv(1,1)=0.d0
ajacinv(1,2)=0.d0
ajacinv(1,3)=(-1.d0 - 2.d0*xin**2 - 2.d0*xin**4 + 2.d0*yin -
! 4.d0*xin**2*yin - 6.d0*yin**2 - 4.d0*xin**2*yin**2 +
! 20.d0*yin**3/3.d0 - 22.d0*yin**4/3.d0)/denom
ajacinv(1,4)=(2.d0*xin + 4.d0*xin**3 + 4.d0*xin*yin +
! 8.d0*xin**3*yin + 4.d0*xin*yin**2 +
! 8.d0*xin*yin**3/3.d0)/denom
ajacinv(2,1)=0.d0
ajacinv(2,2)=0.d0
ajacinv(2,3)=ajacinv(1,4)
ajacinv(2,4)=(-1.d0 - 6.d0*xin**2 - 6.d0*xin**4 - 2.d0*yin -
! 12.d0*xin**2*yin - 2.d0*yin**2 - 12.d0*xin**2*yin**2 -
! 4.d0*yin**3/3.d0 - 2.d0*yin**4/3.d0)/denom
ajacinv(3,1)=1.d0
ajacinv(3,2)=0.d0
ajacinv(3,3)=0.d0
ajacinv(3,4)=0.d0
ajacinv(4,1)=0.d0
ajacinv(4,2)=1.d0
ajacinv(4,3)=0.d0
ajacinv(4,4)=0.d0
return
end

```

Subroutine Taylor(am,h)

```

* This subroutine uses the Taylor method to
* solve the Sixth order truncation of the Toda lattice potential.
double precision am(4),x(0:10),y(0:10),xh,xhd,yh,yhd,h,t
double precision x1,y1,x2,y2
* Set the initial conditions
k=5
x(0)=am(1)
y(0)=am(2)
x(1)=am(3)
y(1)=am(4)
*
x(2)=-x(0) - 2.d0*x(0)**3 - 6.d0*x(0)**5/5.d0 - 2.d0*x(0)*y(0) -
! 4.d0*x(0)**3*y(0) - 2.d0*x(0)*y(0)**2 -
! 4.d0*x(0)**3*y(0)**2 - 4.d0*x(0)*y(0)**3/3.d0 -
! 2.d0*x(0)*y(0)**4/3.d0
y(2)=-x(0)**2 - x(0)**4 - y(0) - 2.d0*x(0)**2*y(0) -
! 2.d0*x(0)**4*y(0) + y(0)**2 - 2.d0*x(0)**2*y(0)**2 -
! 2.d0*y(0)**3 - 4.d0*x(0)**2*y(0)**3/3.d0 + 5.d0*y(0)**4/3.d0 -
! 22.d0*y(0)**5/15.d0
x(3)=-x(1) - 6.d0*x(0)**2*x(1) - 6.d0*x(0)**4*x(1) -
! 2.d0*y(0)*x(1) -

```

```

! 12.d0*x(0)**2*y(0)*x(1) - 2.d0*y(0)**2*x(1) -
! 12.d0*x(0)**2*y(0)**2*x(1) - 4.d0*y(0)**3*x(1)/3.d0 -
! 2.d0*y(0)**4*x(1)/3.d0 -
! 2.d0*x(0)*y(1) - 4.d0*x(0)**3*y(1) - 4.d0*x(0)*y(0)*y(1) -
! 8.d0*x(0)**3*y(0)*y(1) - 4.d0*x(0)*y(0)**2*y(1) -
! 8.d0*x(0)*y(0)**3*y(1)/3.d0
y(3)=-2.d0*x(0)*x(1) - 4.d0*x(0)**3*x(1) - 4.d0*x(0)*y(0)*x(1) -
! 8.d0*x(0)**3*y(0)*x(1) - 4.d0*x(0)*y(0)**2*x(1) -
! 8.d0*x(0)*y(0)**3*x(1)/3.d0 - y(1) - 2.d0*x(0)**2*y(1) -
! 2.d0*x(0)**4*y(1) + 2.d0*y(0)*y(1) - 4.d0*x(0)**2*y(0)*y(1) -
! 6.d0*y(0)**2*y(1) - 4.d0*x(0)**2*y(0)**2*y(1) +
! 20.d0*y(0)**3*y(1)/3.d0 - 22.d0*y(0)**4*y(1)/3.d0
x(4)=-12.d0*x(0)*x(1)**2 - 24.d0*x(0)**3*x(1)**2 -
! 24.d0*x(0)*y(0)*x(1)**2 - 24.d0*x(0)*y(0)**2*x(1)**2 -
! 4.d0*x(1)*y(1) -
! 24.d0*x(0)**2*x(1)*y(1) - 8.d0*y(0)*x(1)*y(1) -
! 48.d0*x(0)**2*y(0)*x(1)*y(1) -
! 8.d0*y(0)**2*x(1)*y(1) - 16.d0*y(0)**3*x(1)*y(1)/3.d0 -
! 4.d0*x(0)*y(1)**2 - 8.d0*x(0)**3*y(1)**2 -
! 8.d0*x(0)*y(0)*y(1)**2 -
! 8.d0*x(0)*y(0)**2*y(1)**2 - x(2) - 6.d0*x(0)**2*x(2) -
! 6.d0*x(0)**4*x(2) - 2.d0*y(0)*x(2) - 12.d0*x(0)**2*y(0)*x(2) -
! 2.d0*y(0)**2*x(2) - 12.d0*x(0)**2*y(0)**2*x(2) -
! 4.d0*y(0)**3*x(2)/3.d0 -
! 2.d0*y(0)**4*x(2)/3.d0 - 2.d0*x(0)*y(2) - 4.d0*x(0)**3*y(2) -
! 4.d0*x(0)*y(0)*y(2) - 8.d0*x(0)**3*y(0)*y(2) -
! 4.d0*x(0)*y(0)**2*y(2) - 8.d0*x(0)*y(0)**3*y(2)/3.d0
y(4)=-2.d0*x(1)**2 - 12.d0*x(0)**2*x(1)**2 - 4.d0*y(0)**2*x(1)**2 -
! 24.d0*x(0)**2*y(0)*x(1)**2 - 4.d0*y(0)**2*x(1)**2 -
! 8.d0*y(0)**3*x(1)**2/3.d0 - 8.d0*x(0)*x(1)*y(1) -
! 16.d0*x(0)**3*x(1)*y(1) - 16.d0*x(0)*y(0)*x(1)*y(1) -
! 16.d0*x(0)*y(0)**2*x(1)*y(1) + 2.d0*y(1)**2 -
! 4.d0*x(0)**2*y(1)**2 - 12.d0*y(0)*y(1)**2 -
! 8.d0*x(0)**2*y(0)*y(1)**2 +
! 20.d0*y(0)**2*y(1)**2 - 88.d0*y(0)**3*y(1)**2/3.d0 -
! 2.d0*x(0)*x(2) -
! 4.d0*x(0)**3*x(2) - 4.d0*x(0)*y(0)*x(2) -
! 8.d0*x(0)**3*y(0)*x(2) -
! 4.d0*x(0)*y(0)**2*x(2) - 8.d0*x(0)*y(0)**3*x(2)/3.d0 - y(2) -
! 2.d0*x(0)**2*y(2) - 2.d0*x(0)**4*y(2) + 2.d0*y(0)*y(2) -
! 4.d0*x(0)**2*y(0)*y(2) - 6.d0*y(0)**2*y(2) -
! 4.d0*x(0)**2*y(0)**2*y(2) +
! 20.d0*y(0)**3*y(2)/3.d0 - 22.d0*y(0)**4*y(2)/3.d0
x1=-12.d0*x(1)**3 - 72.d0*x(0)**2*x(1)**3 -
! 24.d0*y(0)*x(1)**3 - 24.d0*y(0)**2*x(1)**3 -
! 72.d0*x(0)*x(1)**2*y(1) - 144.d0*x(0)*y(0)*x(1)**2*y(1) -
! 12.d0*x(1)*y(1)**2 - 72.d0*x(0)**2*x(1)*y(1)**2 -
! 24.d0*y(0)*x(1)*y(1)**2 - 24.d0*y(0)**2*x(1)*y(1)**2 -
! 8.d0*x(0)*y(1)**3 - 16.d0*x(0)*y(0)*y(1)**3 -
! 36.d0*x(0)*x(1)*x(2) - 72.d0*x(0)**3*x(1)*x(2) -
! 72.d0*x(0)*y(0)*x(1)*x(2) - 72.d0*x(0)*y(0)**2*x(1)*x(2) -
! 6.d0*y(1)*x(2) - 36.d0*x(0)**2*y(1)*x(2) - 6.d0*x(1)*y(2) -
! 12.d0*y(0)*y(1)*x(2) - 72.d0*x(0)**2*y(0)*y(1)*x(2) -
! 12.d0*y(0)**2*y(1)*x(2) - 8.d0*y(0)**3*y(1)*x(2)
x2=-36.d0*x(0)**2*x(1)*y(2) - 12.d0*y(0)*x(1)*y(2) -
! 72.d0*x(0)**2*y(0)*x(1)*y(2) - 12.d0*y(0)**2*x(1)*y(2) -
! 8.d0*y(0)**3*x(1)*y(2) - 12.d0*x(0)*y(1)*y(2) -
! 24.d0*x(0)**3*y(1)*y(2) - 24.d0*x(0)*y(0)*y(1)*y(2) -
! 24.d0*x(0)*y(0)**2*y(1)*y(2) - x(3) - 6.d0*x(0)**2*x(3) -
! 6.d0*x(0)**4*x(3) - 2.d0*y(0)*x(3) - 12.d0*x(0)**2*y(0)*x(3) -
! 2.d0*y(0)**2*x(3) - 12.d0*x(0)**2*y(0)**2*x(3) -
! 4.d0*y(0)**3*x(3)/3.d0 - 2.d0*y(0)**4*x(3)/3.d0 -
! 2.d0*x(0)*y(3) - 4.d0*x(0)**3*y(3) - 4.d0*x(0)*y(0)*y(3) -
! 8.d0*x(0)**3*y(0)*y(3) - 4.d0*x(0)*y(0)**2*y(3) -
! 8.d0*x(0)*y(0)**3*y(3)/3.d0
x(5)=x1+x2
y1=-24.d0*x(0)*x(1)**3 - 48.d0*x(0)*y(0)*x(1)**3 -

```

```

!      12.d0*x(1)**2*y(1) -
!      72.d0*x(0)**2*x(1)**2*y(1) - 24.d0*y(0)*x(1)**2*y(1) -
!      24.d0*y(0)**2*x(1)**2*y(1) - 24.d0*x(0)*x(1)*y(1)**2 -
!      48.d0*x(0)*y(0)*x(1)*y(1)**2 - 12.d0*y(1)**3 -
!      8.d0*x(0)**2*y(1)**3 +
!      40.d0*y(0)*y(1)**3 - 88.d0*y(0)**2*y(1)**3 - 6.d0*x(1)*x(2) -
!      36.d0*x(0)**2*x(1)*x(2) - 12.d0*y(0)*x(1)*x(2) -
!      12.d0*x(0)**2*y(0)*x(1)*x(2) - 12.d0*y(0)**2*x(1)*x(2) -
!      8.d0*y(0)**3*x(1)*x(2) - 12.d0*x(0)*y(1)*x(2) -
!      24.d0*x(0)**3*y(1)*x(2) -
!      24.d0*x(0)*y(0)*y(1)*x(2) - 24.d0*x(0)*y(0)**2*y(1)*x(2) -
!      12.d0*x(0)*x(1)*y(2) - 24.d0*x(0)**3*x(1)*y(2)
y2=-24.d0*x(0)*y(0)*x(1)*y(2) -
!      24.d0*x(0)*y(0)**2*x(1)*y(2) + 6.d0*y(1)*y(2) -
!      12.d0*x(0)**2*y(1)*y(2) - 36.d0*y(0)*y(1)*y(2) -
!      24.d0*x(0)**2*y(0)*y(1)*y(2) + 60.d0*y(0)**2*y(1)*y(2) -
!      88.d0*y(0)**3*y(1)*y(2) - 2.d0*x(0)*x(3) - 4.d0*x(0)**3*x(3) -
!      4.d0*x(0)*y(0)*x(3) - 8.d0*x(0)**3*y(0)*x(3) -
!      4.d0*x(0)*y(0)**2*x(3) -
!      8.d0*x(0)*y(0)**3*x(3)/3.d0 - y(3) - 2.d0*x(0)**2*y(3) -
!      2.d0*x(0)**4*y(3) + 2.d0*y(0)*y(3) - 4.d0*x(0)**2*y(0)*y(3) -
!      6.d0*y(0)**2*y(3) - 4.d0*x(0)**2*y(0)**2*y(3) +
!      20.d0*y(0)**3*y(3)/3.d0 - 22.d0*y(0)**4*y(3)/3.d0
y(5)=y1+y2

```

*

```

xh=0.d0
xhd=0.d0
yh=0.d0
yhd=0.d0
do i=0,k
  xh=xh+x(i)*h**i/factrl(i)
  xhd=xhd+x(i+1)*h**i/factrl(i)
  yh=yh+y(i)*h**i/factrl(i)
  yhd=yhd+y(i+1)*h**i/factrl(i)
end do
am(1)=xh
am(2)=yh
am(3)=xhd
am(4)=yhd
end

```

```

Subroutine derivs(t,x,dxdt)
double precision x(4),dxdt(4),t
dxdt(1)=x(3)
dxdt(2)=x(4)
dxdt(3)=-x(1) - 2.d0*x(1)**3 - 6.d0*x(1)**5/5.d0 -
!      2.d0*x(1)*x(2) - 4.d0*x(1)**3*x(2) - 2.d0*x(1)*x(2)**2 -
!      4.d0*x(1)**3*x(2)**2 - 4.d0*x(1)*x(2)**3/3.d0 -
!      2.d0*x(1)*x(2)**4/3.d0
dxdt(4)=-x(1)**2 - x(1)**4 - x(2) - 2.d0*x(1)**2*x(2) -
!      2.d0*x(1)**4*x(2) + x(2)**2 - 2.d0*x(1)**2*x(2)**2 -
!      2.d0*x(2)**3 - 4.d0*x(1)**2*x(2)**3/3.d0 + 5.d0*x(2)**4/3.d0 -
!      22.d0*x(2)**5/15.d0
return
end

```

```

Subroutine umfld(x1,y1,xdi,ydi,u1,lreg,ii)
Double Precision x(5002),y(5002),z,ajac(4,4),ajacinv(4,4)
Double Precision u1(4),u2(4)
Double Precision xi,yi,xdi,ydi
Double Precision xd(5002),yd(5002)
Double Precision xi,yi,h
Double Precision anorm(4),adent(4,4)
Double Precision theta,csec,dmdt(4),t
double precision delta,E,dimen

```

```

Integer i,k,l,kk,lreg
common // delta,E,dimen
t=0.d0
do i=1,4
  do j=1,4
    adent(i,j)=0.d0
  end do
end do
do i=1,4
  adent(i,i)=1.d0
end do
lreg=0
itime=0
ii=0
iii=0
x(1)=x1
y(1)=y1
xd(1)=xd1
yd(1)=yd1
*
* Set initial choices for the unstable manifold
*
  anorm(1)=1.d0/3.d0
  anorm(2)=1.d0
  anorm(3)=1.d0
  anorm(4)=1.d0
  Call Normal(anorm)
  u1(1)=anorm(1)
  u1(2)=anorm(2)
  u1(3)=anorm(3)
  u1(4)=anorm(4)
  anorm(1)=1.d0
  anorm(2)=1.d0/3.d0
  anorm(3)=1.d0
  anorm(4)=1.d0
  Call Normal(anorm)
  u2(1)=anorm(1)
  u2(2)=anorm(2)
  u2(3)=anorm(3)
  u2(4)=anorm(4)
  h=1.d-2
*
* Make sure that the two choices for the unstable manifold converge.
*
  i=1
100      xi=x(i)
          yi=y(i)
          Call Jacob(ajac,xi,yi)
          do k=1,4
            do j=1,4
              ajac(k,j)=ajac(k,j)*h+adent(k,j)
            end do
            anorm(k)=ajac(k,1)*u1(1)+ajac(k,2)*u1(2)+
!              ajac(k,3)*u1(3)+ajac(k,4)*u1(4)
            end do
            Call Normal(anorm)
            u1(1)=anorm(1)
            u1(2)=anorm(2)
            u1(3)=anorm(3)
            u1(4)=anorm(4)
            do k=1,4
              anorm(k)=ajac(k,1)*u2(1)+ajac(k,2)*u2(2)+
!              ajac(k,3)*u2(3)+ajac(k,4)*u2(4)
            end do
            Call Normal(anorm)
            u2(1)=anorm(1)
            u2(2)=anorm(2)

```

```

u2(3)=anorm(3)
u2(4)=anorm(4)
anorm(1)=x(i)
anorm(2)=y(i)
anorm(3)=xd(i)
anorm(4)=yd(i)
call Taylor(anorm,h)
x(i+1)=anorm(1)
y(i+1)=anorm(2)
xd(i+1)=anorm(3)
yd(i+1)=anorm(4)
if ((dabs(x(i+1)) .gt. 1.d03) .or.
    (dabs(y(i+1)) .gt. 1.d03)) then
!
    lreg = 2
    write(i9,*)x(i+1),y(i+1)
    return
endif
call angle(u1,u2,csec,theta)
if (Dsin(theta) .lt. 1.d-6)then
    ii=i
    xi=x(i)
    yi=y(i)
    xd1=xd(i)
    yd1=yd(i)
    return
endif
i=i+1
if (i .gt. 5000) then
    lreg = 1
    return
endif
goto100
end

```

```

Subroutine smfld(x1,y1,xd1,yd1,s1,lreg,ii)
Double Precision x(20001),y(20001),z,ajac(4,4),ajacinv(4,4)
Double Precision s1(4),s2(4)
Double Precision x1,y1,xd1,yd1
Double Precision xd(20001),yd(20001)
Double Precision xi,yi,h
Double Precision anorm(4),adent(4,4)
Double Precision theta,csec,dmdt(4)
double precision delta,E,dimen
Integer i,k,l,kk,lreg
common // delta,E,dimen
do i=1,4
    do j=1,4
        adent(i,j)=0.d0
    end do
end do
do i=1,4
    adent(i,i)=1.d0
end do
if ((lreg .eq. 1) .or. (lreg .eq.2)) then
    return
endif
lreg=0
iii=0
x(ii)=x1
y(ii)=y1
xd(ii)=xd1
yd(ii)=yd1
do i=ii,ii+5000
    anorm(1)=x(i)
    anorm(2)=y(i)
    anorm(3)=xd(i)

```

```

        anorm(4)=yd(i)
        h=1.d-4
        call Taylor(anorm,h)
        x(i+1)=anorm(1)
        y(i+1)=anorm(2)
        xd(i+1)=anorm(3)
        yd(i+1)=anorm(4)
        if ((dabs(x(i+1))) .gt. 1.d03) .or.
!           (dabs(y(i+1))) .gt. 1.d03) then
                lreg = 2
                write(19,*)x(i+1),y(i+1)
                return
        endif
    end do
do i=ii+5000,20000
    anorm(1)=x(i)
    anorm(2)=y(i)
    anorm(3)=xd(i)
    anorm(4)=yd(i)
    h=1.d-2
    call Taylor(anorm,h)
    x(i+1)=anorm(1)
    y(i+1)=anorm(2)
    xd(i+1)=anorm(3)
    yd(i+1)=anorm(4)
!   if ((dabs(x(i+1))) .gt. 1.d03) .or.
       (dabs(y(i+1))) .gt. 1.d03) then
            lreg = 2
            write(19,*)x(i+1),y(i+1)
            return
    endif
end do

*
* Approximate the stable direction by some random vector.
*
    i=20000
250  anorm(1)=1.d0
    anorm(2)=1.d0
    anorm(3)=1.d0/3.d0
    anorm(4)=1.d0
    Call Normal(anorm)
    s1(1)=anorm(1)
    s1(2)=anorm(2)
    s1(3)=anorm(3)
    s1(4)=anorm(4)
    anorm(1)=1.d0
    anorm(2)=1.d0
    anorm(3)=1.d0
    anorm(4)=1.d0/3.d0
    Call Normal(anorm)
    s2(1)=anorm(1)
    s2(2)=anorm(2)
    s2(3)=anorm(3)
    s2(4)=anorm(4)

*
* Make sure that the two choices for the stable manifold converge.
*
300  xi=x(i)
    yi=y(i)
    Call Jacinv(ajacinv,xi,yi)
    do k=1,4
        do j=1,4
            ajacinv(k,j)=ajacinv(k,j)*h+adent(k,j)
        end do
        anorm(k)=ajacinv(k,1)*s1(1)+ajacinv(k,2)*s1(2)+
!           ajacinv(k,3)*s1(3)+ajacinv(k,4)*s1(4)
    end do

```

```

      Call Normal(anorm)
      s1(1)=anorm(1)
      s1(2)=anorm(2)
      s1(3)=anorm(3)
      s1(4)=anorm(4)
do k=1,4
      anorm(k)=ajacinv(k,1)*s2(1)+ajacinv(k,2)*s2(2)+
!          ajacinv(k,3)*s2(3)+ajacinv(k,4)*s2(4)
end do
      Call Normal(anorm)
      s2(1)=anorm(1)
      s2(2)=anorm(2)
      s2(3)=anorm(3)
      s2(4)=anorm(4)
      call angle(s1,s2,csec,theta)
      if (Dsin(theta) .lt. 1.d-6) then
          iii=i
          goto 400
      endif
i=i-1
if (i .eq. ii+5000) then
    lreg=1
    write(6,*)'lreg=',lreg
    return
endif
goto 300
400 do i=iii,ii+5000,-1
      xi=x(i)
      yi=y(i)
      Call Jacinv(ajacinv,xi,yi)
do k=1,4
      do j=1,4
          ajacinv(k,j)=ajacinv(k,j)*h+adent(k,j)
      end do
      anorm(k)=ajacinv(k,1)*s1(1)+ajacinv(k,2)*s1(2)+
!          ajacinv(k,3)*s1(3)+ajacinv(k,4)*s1(4)
end do
      Call Normal(anorm)
      s1(1)=anorm(1)
      s1(2)=anorm(2)
      s1(3)=anorm(3)
      s1(4)=anorm(4)
end do
x1=x(ii)
y1=y(ii)
xd1=xd(ii)
yd1=yd(ii)
Return
end

```

APPENDIX B

FORTRAN CODES - FOUR-DIMENSIONAL SYSTEMS

```

Program Shadow
Double precision x1,y1,rand1
Integer i,ii,kk,jran,lr
open (unit=18,file='short.dat',status='new')
open (unit=19,file='short2.dat',status='new')
jran=3037
do i=0,9
  Call rand1(jran)
  x1=rand1(jran)*1.D-14
  Call rand1(jran)
  y1=rand1(jran)*1.d-14
  write(18,*)x1,y1
end do
rewind (unit=18)
do ii=1,10
  kk=0
  lr=0
  lr=1
  read(18,*)x1,y1
  call mflld(x1,y1,xd1,yd1,ll,lr)
  write(19,*)x1,y1,ll
  if (lr .eq. 1) then
    write(16,*) 'Point is within regular region'
  endif
  if (lr .eq. 2) then
    write(16,*) 'Particle escapes'
  endif
end do
end
Subroutine mflld(x1,y1,xd1,yd1,ll,lr)
Double Precision x(6000),y(6000),z,J(2,2),Jinv(2,2),m(2)
Double Precision sone(2,6000),stwo(2,6000),csc(6000),csec
Double Precision u1(2),u2(2),s1(2),s2(2),rn(6000),tn(6000)
Double Precision cn(6000),dn(6000),bnd,xin,yin,xout,yout
Double Precision abnd,en(6000),csc1,csc2,rndum,tndum,cdum,ddum
Double Precision delta,bnd,a,b,x1,y1,xd1,yd1,mdim,s(2,6000)
Double Precision uone(2,6000),utwo(2,6000),udiff1,udiff2
Double Precision xdin,xdout,ydin,ydout,xdot(6000),ydot(6000)
Integer N,i,k,l,kk,ll,lr
delta=1.0d-14
a=1.4d0
b=0.3d0
N=6000
mdim=2.d0
lr=0
ll=0
bnd=0.d0
x(1)=x1
y(1)=y1
xdot(1)=xd1

```

```

ydot(1)=yd1
m(1)=dsqrt(2.d0)
m(2)=1.d0
Call Normal(m)
do jj=1,2
  uone(jj,1)=m(jj)
end do
m(1)=b/3.d0
m(2)=1.d0
Call Normal(m)
do jj=1,2
  utwo(jj,1)=m(jj)
end do
cn(1)=0.d0
abnd=mdim*(-5.d0/2.d0)*4.d0*(-2.d0)/dSqrt(delta)
dn(N)=abnd
en(1)=dn(N)
l=0
Do k=1,20000
Do i=1,N
  xin=x(i)
  yin=y(i)
  Call Jacob(J,a,b,xin,yin)
  do jj=1,2
    m(jj)=J(jj,1)*uone(1,i)+J(jj,2)*uone(2,i)
  end do
  Call Normal(m)
  do jj=1,2
    uone(jj,i+1)=m(jj)
  end do
  do jj=1,2
    m(jj)=J(jj,1)*utwo(1,i)+J(jj,2)*utwo(2,i)
  end do
  do jj=1,2
    m(jj)=m(jj)-uone(jj,i+1)*(uone(1,i+1)*m(1)+uone(2,i+1)*m(2))
  end do
  Call Normal(m)
  do jj=1,2
    utwo(jj,i+1)=m(jj)
  end do
  xin=x(i)
  yin=y(i)
  xdin=xdot(i)
  ydin=ydot(i)
  call map(xin,yin,xdin,ydin,xout,yout,xdout,ydout,a,b)
  x(i+1)=xout
  y(i+1)=yout
  xdot(i+1)=xdout
  ydot(i+1)=ydout
  if ((dabs(x(i+1)) .gt. 1.d03) .or. (dabs(y(i+1)) .gt. 1.d03))
then
    lr = 2
    return
  endif
  bnd=dmax1(bnd,Dabs(x(i)),Dabs(y(i)))
End do
bbnd=2.d0*bnd
m(1)=dsqrt(2.d0)
m(2)=1.d0
Call Normal(m)
do jj=1,2
  sone(jj,N+1)=m(jj)
end do
m(1)=1.d0
m(2)=dsqrt(2.d0)
Call Normal(m)
do jj=1,2

```

```

      stwo(jj,N+1)=m(jj)
    end do
  Do i=N,1,-1
    xin=x(i)
    yin=y(i)
    Call Jacinv(Jinv,a,b,xin,yin)
    do jj=1,2
      m(jj)=Jinv(jj,1)*sone(1,i+1)+Jinv(jj,2)*sone(2,i+1)
    end do
    Call Normal(m)
    do jj=1,2
      sone(jj,i)=m(jj)
    end do
    do jj=1,2
      m(jj)=Jinv(jj,1)*stwo(1,i+1)+Jinv(jj,2)*stwo(2,i+1)
    end do
    do jj=1,2
      m(jj)=m(jj)-sone(jj,i)*(sone(1,i)*m(1)+sone(2,i)*m(2))
    end do
    Call Normal(m)
    do jj=1,2
      stwo(jj,i)=m(jj)
    end do
  End do
  Do i=1,5000
    do jj=1,2
      u1(jj)=uone(jj,i)
      u2(jj)=utwo(jj,i)
      s1(jj)=sone(jj,i)
      s2(jj)=stwo(jj,i)
    end do
    Call angle(u1,u2,s1,s2,csec)
    csc(i)=csec
    do jj=1,2
      u1(jj)=uone(jj,i+1)
      s1(jj)=sone(jj,i+1)
      u2(jj)=utwo(jj,i+1)
      s2(jj)=stwo(jj,i+1)
    end do
    Call angle(u1,u2,s1,s2,csec)
    csc(i+1)=csec
    xin=x(i)
    yin=y(i)
    Call Jacob(J,a,b,xin,yin)
    Call Jacinv(Jinv,a,b,xin,yin)
    do jj=1,2
      u1(jj)=uone(jj,i)
      u2(jj)=utwo(jj,i)
      s1(jj)=sone(jj,i)
      s2(jj)=stwo(jj,i)
    end do
    Call bound(J,Jinv,u1,u2,s1,s2,rdum,tdum)
    rn(i)=rdum
    tn(i)=tdum
    cdum=cn(i)
    ddum=en(i)
    csc1=csc(i)
    csc2=csc(i+1)
    Call recur(csc1,csc2,rdum,tdum,cdum,ddum,abnd,k,kk,1,11)
    if (1 .eq. 1) return
    cn(i+1)=cdum
    en(i+1)=ddum
  End do
  x(1)=x(5001)
  y(1)=y(5001)
  xdot(1)=xdot(5001)
  ydot(1)=ydot(5001)

```

```

en(1)=en(5001)
cn(1)=cn(5001)
end do
Write(6,*) 'map can be shadowed for over 100 million iterates.'
End
Subroutine Normal(m)
Double Precision m(2),w
w=DSqrt(m(1)**2+m(2)**2)
m(1)=m(1)/w
m(2)=m(2)/w
Return
End
Subroutine angle(au,au2,as,as2,csec)
Double Precision au(2),as(2),csec,atm,atn
Double Precision an(2,2),bn(2,2),bnu(2,2),bns(2,2)
Double Precision sn(2,2),un(2,2),id(2,2)
double Precision csec1,csec2,csecdif
Double Precision a(2,2),w(2),fv1(2),fv2(2)
n=2
np=2
do jj=1,2
  an(jj,1)=as(jj)
  an(jj,2)=au(jj)
end do
Call matinv(an,bn,n,np,indx)
do jj=1,2
  do jjj=1,2
    bns(jj,jjj)=0.d0
    bnu(jj,jjj)=0.d0
  end do
end do
do jj=1,2
  bns(1,jj)=bn(1,jj)
  bnu(2,jj)=bn(2,jj)
end do
do ii=1,2
  do jj=1,2
    sn(ii,jj)=as(ii)*bns(1,jj)
    un(ii,jj)=au(ii)*bnu(2,jj)
  end do
end do
do ii=1,2
  do jj=1,2
    id(ii,jj)=sn(ii,jj)+un(ii,jj)
  end do
end do
do ii=1,2
  id(ii,ii)=id(ii,ii)-1.d0
end do
do ii=1,2
  do jj=1,2
    if (id(ii,jj) .gt. 1.d-5) then
      write(6,*) 'error caused in matrix inversion'
      stop
    end if
  end do
end do
do ii=1,2
  do jj=1,2
    a(ii,jj)=sn(1,ii)*sn(1,jj)+sn(2,ii)*sn(2,jj)
  end do
end do
call tred1(np,n,a,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0) then
  write(6,*) 'error caused in eigenvalue routine'
  stop

```

```

end if
csec1=Dsqrt(max(w(1),w(2)))
do ii=1,2
  do jj=1,2
    a(ii,jj)=un(1,ii)*un(1,jj)+un(2,ii)*un(2,jj)
  end do
end do
call tred1(np,n,a,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0) then
  write(6,*)'error caused in eigenvalue routine'
  stop
end if
csec2=Dsqrt(max(w(1),w(2)))
csecdif=dabs(csec1-csec2)
if (csecdif .gt. 1.d-5) then
  write(6,*)'error caused in matrix inversion'
  stop
endif
csec=csec1
Return
End
Subroutine bound(J,Jinv,u1,u2,s1,s2,rndum,tndum)
Double precision J(2,2),Jinv(2,2),u1(2),u2(2),s1(2),s2(2)
Double precision sj(2,2),uj(2,2),sjt(2,2),ujt(2,2)
Double precision rndum,tndum,sigma1,sigma2,s(2,2),u(2,2)
Double precision id(2,2),a(2,2),b(2,2),w(2),fv1(2),fv2(2)
np=2
n=2
u2(1)=0.d0
u2(2)=0.d0
s2(1)=0.d0
s2(2)=0.d0
do ii=1,2
  s(ii,1)=s1(ii)
  s(ii,2)=s2(ii)
  u(ii,1)=u1(ii)
  u(ii,2)=u2(ii)
end do
do ii=1,2
  do jj=1,2
    sj(ii,jj)=J(ii,1)*s(1,jj)+J(ii,2)*s(2,jj)
    uj(ii,jj)=Jinv(ii,1)*u(1,jj)+Jinv(ii,2)*u(2,jj)
  end do
end do
do ii=1,2
  do jj=1,2
    sjt(ii,jj)=sj(1,ii)*sj(1,jj)+sj(2,ii)*sj(2,jj)
    ujt(ii,jj)=uj(1,ii)*uj(1,jj)+uj(2,ii)*uj(2,jj)
  end do
end do
call tred1(np,n,sjt,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0) then
  write(6,*)'error caused in eigenvalues in subroutine bound'
  stop
end if
sigma1=Dsqrt(max(w(1),w(2)))
do ii=1,2
  do jj=1,2
    id(ii,jj)=0.d0
    id(1,1)=1.d0
  end do
end do
do ii=1,2
  do jj=1,2
    sj(ii,jj)=s(1,ii)*s(1,jj)+s(2,ii)*s(2,jj)-id(ii,jj)
  end do
end do

```

```

        end do
    end do
do ii=1,2
    do jj=1,2
        sjt(ii,jj)=sj(1,ii)*sj(1,jj)+sj(2,ii)*sj(2,jj)
    end do
end do
call tred1(np,n,sjt,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0)then
    write(6,*)'error caused in eigenvalues in subroutine bound'
    stop
end if
sigma2=Dsqrt(max(w(1),w(2)))
rndum=sigma1/dsqrt(1.d0-sigma2)
call tred1(np,n,ujt,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0)then
    write(6,*)'error caused in eigenvalues in subroutine bound'
    stop
end if
sigma1=Dsqrt(max(w(1),w(2)))
do ii=1,2
    do jj=1,2
        uj(ii,jj)=u(1,ii)*u(1,jj)+u(2,ii)*u(2,jj)-id(ii,jj)
    end do
end do
do ii=1,2
    do jj=1,2
        ujt(ii,jj)=uj(1,ii)*uj(1,jj)+uj(2,ii)*uj(2,jj)
    end do
end do
call tred1(np,n,ujt,w,fv1,fv2)
call tqlrat(n,w,fv2,ierr)
if (ierr .ne. 0)then
    write(6,*)'error caused in eigenvalues in subroutine bound'
    stop
end if
sigma2=Dsqrt(max(w(1),w(2)))
tndum=sigma1/dsqrt(1.d0-sigma2)
Return
End
Subroutine recur(csc1,csc2,rndum,tndum,cdum,edum,abnd,k,kk,1,ll)
Double precision csc1,csc2,abnd
Double precision rndum,tndum,cdum,edum
ll=ll+1
cdum=csc2+rndum*cdum
edum=dmin1((edum-csc1)/tndum,abnd)
if ((cdum.gt. abnd).or. (edum.lt. 0.0)) then
    write(6,*) 'unstable point is no longer shadowable'
    write(6,*) 'length is approximately ',k*5000,' iterates.'
    write(6,*) ll
    l=1
    kk=k*5000
endif
return
End
Subroutine Jacob(J,a,b,xin,yin)
Double Precision J(2,2),a,b,xin,yin
J(1,1)=-2.d0*xin
J(1,2)=b
J(2,1)=1.d0
J(2,2)=0.d0
return
end
Subroutine Jacinv(Jinv,a,b,xin,yin)
Double Precision Jinv(2,2),a,b,xin,yin

```

```
Jinv(1,1)=0.d0
Jinv(1,2)=1.d0
Jinv(2,1)=1.d0/b
Jinv(2,2)=2.d0*yin/b
return
end
Subroutine map(xin,yin,xdin,ydin,xout,yout,xdout,ydout,a,b)
Double Precision xin,yin,xout,yout,a,b
xout=a-xin*xin+b*yin
yout=xin
return
End
```

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BIOGRAPHICAL SKETCH

David Eric Willmes was born in a suburb of Chicago in 1965. Having graduated from East Grand Rapids High School in Michigan in 1983, he attended Lehigh University as an engineering student for two years. A course in quantum mechanics that he took as an elective led to his decision to change his major to physics. In 1985, he transferred to Washington University in St. Louis, where he graduated with a degree in physics in 1987.

Dave's graduate career began at the University of Kansas, where he received his master's degree in physics in 1989. He then attended the University of Florida, where he will remain for the next six and a half years to earn his doctorate.

While not cogitating upon the mysteries of the universe and studying physics, Dave enjoys skiing (even after two broken bones), cave diving, and playing his bass.

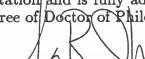
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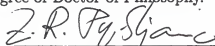
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